Point-wise and Uniform Convergence of Fourier Series

Mevan Wijewardena

Abstract

In this article, we explore the problems of point-wise and uniform Convergence of the Fourier Series, using the knowledge on the mean-square convergence of the Fourier series.

I. INTRODUCTION

In this article, we consider the problems of point-wise and uniform convergence of the Fourier series. We assume the knowledge on the mean-square converge of Fourier series for the 2π-periodic complex-valued functions, which are Riemann integrable in $[-\pi, \pi]$ [1]. However, it should be noted that the point-wise convergence of the Fourier series is not guaranteed for all such functions. Nevertheless, point-wise and uniform convergence can be guaranteed under additional conditions on the function considered. For instance, if the function considered is Lipschitz continuous, it can be established that the Fourier series not only converges point-wise but also converges uniformly to the function considered. Also, it can be established that the Fourier series converges point-wise to the function at the points at which the function is differentiable. The continuity of the function does not, in general, guarantee the point-wise convergence of the Fourier series. In summary, we will establish that for a 2π-periodic complex-valued Riemann integrable function,

- the Fourier series converges uniformly to the function if the function is Lipschitz continuous.
- the Fourier series converges point-wise to the function at the points at which the function is differentiable
- the Fourier series does not, in general, converge point-wise if the function is continuous. In particular, we will prove the existence of a continuous function whose Fourier series diverges at 0.

Although point-wise convergence of the Fourier series is not guaranteed for continuous functions, we will establish that the Fourier series is Césaro summable to the function at the points of continuity.

II. Notation

Let $\mathbb{C}^{[-\pi,\pi]}$ denote the space of 2π periodic complex valued functions defined on $\mathbb{R}$. It should be noted that $\mathbb{C}^{[-\pi,\pi]}$ is isomorphic to the space of complex-valued functions defined on the unit circle $S^1 = \{z \in \mathbb{Z} : \ |z| = 1\}$. Also $L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$, $B([-\pi,\pi],\mathbb{C})$, $C([-\pi,\pi],\mathbb{C})$, and $C^m([-\pi,\pi],\mathbb{C})$ denote the spaces of locally Riemann integrable, bounded, continuous, and m-times continuously differentiable ($m \in \mathbb{N}$) 2π-periodic complex-valued functions defined on $\mathbb{R}$, respectively. Additionally $e_n \in \mathbb{C}^{[-\pi,\pi]}$ is defined by $e_n(\theta) = e^{in\theta}$ for $\theta \in \mathbb{R}$. Also $P_{\text{trig}}([-\pi,\pi];\mathbb{C})$ denotes the set of functions of the form,

$$f = \sum_{n=-N}^{N} c_n e_n, \tag{1}$$

where $c_n \in \mathbb{C}$, and $-N \leq n \leq N$.

Additionally, $f(x) = \mathcal{O}(g(x))$ as $x \to a$ indicates that, there exists a neighborhood U of a and a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \in U$. For a normed vector space X with norm $\|.\|_X$, $x \in X$ and $r > 0$, $B_X(x,r)$ denotes the open ball centered at x with radius r i.e $B_X(x,r) = \{y \in X : \ \|x - y\|_X < r\}$. Also, for two normed vector spaces X and Y, $\mathcal{B}(X,Y)$ denotes the set of all bounded linear operators from X to Y.

III. Preliminaries

Proofs for all the results not proved in this section can be found in [1].

A. *Some results regarding $L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$ space*

Notice that $L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$ is isomorphic to the space of Riemann integrable functions defined on $[-\pi,\pi]$ taking equal values at $-\pi$ and π. We can define an inner product $\langle \cdot, \cdot \rangle$ on $L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$ using,

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)} \mathrm{d}\theta. \tag{2}$$

Also, this induces the L^2 norm $\|\cdot\|$,

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 \mathrm{d}\theta. \tag{3}$$

However, for the above definitions to be valid, we should interpret $f = g$ as f and g agree almost everywhere.

More generally, we can define the L^p norm on for $f \in L^2_{\mathcal{R}}([-\pi, \pi]; \mathbb{C})$, where $p \geq 1$ by,

$$\|f\|_p^p = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^p \mathrm{d}\theta. \tag{4}$$

Also the uniform norm $||.||_u$ (also denoted by $\|.\|_\infty$) can be defined for $f \in B([-\pi, \pi]; \mathbb{C})$ by, $||f||_u = \sup_{x \in [-\pi, \pi]} |f(x)|$. We prove the following important result regarding the uniform norm and the L^2 norm.

Lemma 1. *The uniform norm is stronger than the L^2 norm in $L^2_{\mathcal{R}}([-\pi, \pi]; \mathbb{C})$.*

Proof. Notice that,

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 \mathrm{d}\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} ||f||_u^2 \mathrm{d}\theta = ||f||_u^2, \tag{5}$$

which establishes the result. □

B. Fourier Series

Given $f \in \mathbb{C}^{[-\pi, \pi]}$ the series,

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n, \tag{6}$$

is the Fourier series of f, where,

$$\hat{f}(n) = \langle f, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \overline{e_n(\theta)} \mathrm{d}\theta \tag{7}$$

We define the N-th partial sum of the series by $S_N(f) \in P_{\text{trig}}([-\pi, \pi]; \mathbb{C})$, where

$$S_N(f) = \sum_{n=-N}^{N} \hat{f}(n) e_n. \tag{8}$$

The convergence of the Fourier series is defined by the convergence of the partial sums $S_N(f)$.

We assume the knowledge of the convergence of the Fourier series in the mean square sense. We summarize this and a few other results below.

Theorem 1. *Consider $f \in L^2_{\mathcal{R}}([-\pi, \pi]; \mathbb{C})$. Then we have,*

$$\lim_{N \to \infty} ||f - S_N(f)||_2 = 0. \tag{9}$$

Lemma 2. *Consider* $f \in C^1([-\pi,\pi];\mathbb{C})$. *Then we have,*

$$\hat{f}'(n) = in\hat{f}(n), \tag{10}$$

for all $n \in \mathbb{Z}$.

Proof. For $n \neq 0$,

$$\begin{aligned}\hat{f}'(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(y)e^{-iny}\mathrm{d}y =_{(a)} \left[f(y)e^{-iny}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(y)(-in)e^{-iny}\mathrm{d}y \\ &=_{(b)} \frac{in}{2\pi}\int_{-\pi}^{\pi} f'(y)e^{-iny}\mathrm{d}y = in\hat{f}(n).\end{aligned} \tag{11}$$

where (a) follows from integration by parts formulae, and (b) follows since both f and e_{-n} are 2π-periodic. For $n = 0$,

$$\hat{f}'(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(y)\mathrm{d}y =_{(a)} f(\pi) - f(-\pi) =_{(b)} 0. \tag{12}$$

where (a) follows from the Fundamental theorem of Calculus, and (b) follows since f is 2π-periodic. ☐

Theorem 2. *(Parseval's identity for Fourier Series): Consider* $f, g \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$. *Then,*

$$\langle f, g\rangle = \left\langle (\hat{f}(n))_{n=-\infty}^{\infty}, (\hat{g}(n))_{n=-\infty}^{\infty}\right\rangle_{l^2(\mathbb{Z};\mathbb{C})} = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}. \tag{13}$$

In particular,

$$||f||_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \tag{14}$$

Lemma 3. *(Riemann–Lebesgue Lemma for Fourier Series): Consider* $f \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$. *Then,*

$$\lim_{|n|\to\infty} \hat{f}(n) = 0. \tag{15}$$

An equivalent formulation is,

$$\lim_{|n|\to\infty} \int_{-\pi}^{\pi} f(y)\sin(ny)dy = 0, \tag{16}$$

and

$$\lim_{|n|\to\infty} \int_{-\pi}^{\pi} f(y)\cos(ny)dy = 0 \tag{17}$$

C. Approximate Identities

A family of functions $\{f_n\}_{n=1}^{\infty}$ of $L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$ is called an approximate identity if each of the following three conditions is satisfied.

1) For each $n \in \mathbb{N}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(x)\mathrm{d}x = 1, \tag{18}$$

2) There exists $B \geq 0$ such that,

$$\int_{-\pi}^{\pi} |f_n(x)|\mathrm{d}x \leq B, \tag{19}$$

for each $n \in \mathbb{N}$.

3) Given any $\delta \in (0,\pi)$, we have that,

$$\lim_{n\to\infty}\left(\int_{-\pi}^{-\delta} |f_n(x)|\mathrm{d}x + \int_{\delta}^{\pi} |f_n(x)|\mathrm{d}x\right) = 0 \tag{20}$$

D. Convolutions

Consider two functions, $f, g \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$, The convolution of f and g denoted by $f * g \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$ is a function defined by,

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\mathrm{d}y \tag{21}$$

The following lemma summarizes several results regarding convolutions.

Lemma 4. *For $f, g, h \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$, we have that,*

1) $(f * g) \in C([-\pi,\pi],\mathbb{C})$
2) $f * g = g * f$
3) *For* $c \in \mathbb{C}$, $(cf + g) * h = c(f * h) + (g * h)$
4) $||f * g||_u \leq ||f||_u ||g||_1$

We also have the following theorem regarding convolutions and approximate identities.

Theorem 3. *Given an approximate identity $\{f_n\}_{n=1}^{\infty}$ and $f \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$. If f is continuous at x, then,*

$$\lim_{n\to\infty}(f_n * f)(x) = f(x), \tag{22}$$

Moreover, if f is continuous in (a,b), then $f_n * f \to f$ uniformly in any compact sub-interval $[c,d]$ contained in (a,b)

E. Césaro Summability

Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$ and define the k-th partial sum by,

$$s_k = \sum_{i=1}^{k} a_i. \tag{23}$$

The sequence $(a_n)_{n=1}^{\infty}$ (or the sequence of partial sums $(s_n)_{n=1}^{\infty}$) is Césaro summable with Césaro sum A if and only if,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k = A. \tag{24}$$

It can be proved that if the sequence of partial sums s_n converges to A, it is Césaro summable to A.

F. Dirichlet Kernel

It should be noted that for $f \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$, $S_N(f)$ can be written as the convolution of f with another function. In particular, notice that

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} \mathrm{d}y \right) e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^{N} e^{in(x-y)} \right) \mathrm{d}y = (f * D_N)(x), \end{aligned} \tag{25}$$

where,

$$D_N = \sum_{n=-N}^{N} e_n \tag{26}$$

is called the Dirichlet kernel. This representation of the Fourier series will be important later in our analysis. For now, we will use this representation to prove the following results regarding the Fourier series.

Lemma 5. *Consider $N \in \mathbb{N}$, and $f_1, f_2 \in C([-\pi,\pi];\mathbb{C})$, $c \in \mathbb{C}$. Then we have,*

$$S_N(f_1 + cf_2) = S_N(f_1) + cS_N(f_2) \tag{27}$$

Proof. Notice that for $x \in \mathbb{R}$,

$$S_N(f_1 + cf_2) = D_N * (f_1 + cf_2) =_{(a)} D_N * f_1 + c(D_N * f_2) = S_N(f_1) + cS_N(f_2), \tag{28}$$

where we have used lemma 4-1) and lemma 4-3) for (a). □

The following lemma establishes a few properties of the Dirichlet kernel.

Lemma 6. *Following are a few properties of the Dirichlet kernel.*

1) *For all $N \geq 0$, and $y \in \mathbb{R}$ we have,*

$$D_N(y) = \frac{\sin\left(\left(N + \frac{1}{2}\right) y\right)}{\sin\left(\frac{y}{2}\right)} \tag{29}$$

In particular, D_N is real-valued and even.

2) *The following identity holds,*

$$\int_{-\pi}^{\pi} D_N(y) dy = 2\pi \tag{30}$$

3) *The following inequality holds for each $N \geq 2$.*

$$||D_N||_1 \geq \frac{4}{\pi^2} \sum_{k=2}^{N} \frac{1}{k} \tag{31}$$

Proof. See appendix A □

IV. Point-wise and Uniform Convergence of Fourier Series.

The problem of point-wise convergence of the Fourier series can be formulated as follows. Consider a function $f \in L^2_{\mathcal{R}}([-\pi, \pi]; \mathbb{C})$ and $x \in \mathbb{R}$. Under what conditions does $\lim_{N\to\infty} S_N(f)(x)$ exist and is finite? An extension of the above question would be to find the cases in which the above is valid for all x. If this were true, one could find a function $g \in L^2_{\mathcal{R}}([-\pi, \pi]; \mathbb{C})$ such that $\lim_{N\to\infty} S_N(f)(x) = g(x)$ for all $x \in \mathbb{R}$. When such a function exists, the problem of uniform convergence of the Fourier Series is to determine whether $\lim_{n\to\infty} ||S_N(f) - g||_u = 0$. Notice that if point-wise convergence does not happen at all real numbers, we cannot talk about uniform convergence.

Now that we have formally defined the problems of point-wise and uniform convergence of the Fourier series, we will explore the different cases under which these conditions hold. Indeed, it can be established that, for general functions in $L^2_{\mathcal{R}}([-\pi, \pi]; \mathbb{C})$, the point-wise convergence of Fourier series is not guaranteed. Although intuition suggests that the point-wise convergence is true for continuous functions, it is possible to construct $C([-\pi, \pi]; \mathbb{C})$ functions for which the Fourier series diverges at a point.

We begin with the following theorem, which serves as a sufficient condition for determining whether the Fourier series converges uniformly.

Theorem 4. *Given* $f \in C([-\pi,\pi];\mathbb{C})$. *If,*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty, \tag{32}$$

then $S_N(f)$ *converges to* f *uniformly in* $[-\pi,\pi]$.

Proof. Notice that for $n > 0$,

$$|\hat{f}(n)e^{in\theta} + \hat{f}(-n)e^{-in\theta}| \leq |\hat{f}(n)| + |\hat{f}(-n)|. \tag{33}$$

Hence, the Weierstrass M-Test, applied to the sequence of functions, $\{\hat{f}(0)e_0\} \cup \{\hat{f}(n)e_n + \hat{f}(-n)e_{-n}\}_{n=1}^{\infty}$ yields that, $S_N(f)$ converges uniformly to some function g. But notice that $S_N(f)$ is continuous for each N. Hence from the uniform limit theorem, g is continuous. Also $g \in C([-\pi,\pi];\mathbb{C})$, since $S_N(f) \in C([-\pi,\pi];\mathbb{C})$ for each N.

Now we prove that $f \equiv g$. Fix any $\varepsilon > 0$. Notice that, from the mean square convergence of the Fourier Series (theorem 1), there exists $N \in \mathbb{N}$ such that $n \geq N$ implies,

$$||S_n(f) - f|| < \frac{\varepsilon}{2} \tag{34}$$

But notice that since $S_n(f)$ uniformly converges to g, there exists, $M \in \mathbb{N}$ such that $m \geq M$ implies,

$$||S_m(f) - g|| \leq ||S_m(f) - g||_u < \frac{\varepsilon}{2}, \tag{35}$$

where the first inequality follows from lemma 1. Let $\tilde{N} = \max\{N, M\}$. Hence, we have

$$\varepsilon > ||S_{\tilde{N}}(f) - g|| + ||f - S_{\tilde{N}}(f)|| \geq ||g - f||, \tag{36}$$

where the last inequality follows from the triangle inequality. But this will be true for any $\varepsilon > 0$. Hence $||g - f|| = 0$. Since g and f are continuous, we have that $g = f$. □

In the following two sections, we will analyze two cases in which the uniform convergence and the point-wise convergence of the Fourier series hold, respectively. In the section that follows, we will prove the existence of a continuous function for which the Fourier series diverges at 0. Finally, we will explore the problem of Césaro summability of the Fourier series of continuous functions.

A. Uniform Convergence of the Fourier Series of Lipschitz Functions

In this section, we establish that if $f \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$, and f is Lipschitz continuous ($|f(x)-f(y)| \leq L|x-y|$ for all $x, y \in \mathbb{R}$ for some constant $L > 0$), then $S_N(f)$ converges uniformly to f. We present this in the following theorem.

Theorem 5. *Consider $f \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$. Moreover, assume that f is Lipschitz continuous. Then $S_N(f)$ converges uniformly to f.*

Proof. This proof follows the outline provided in Chapter-3, Exercise-16 in [2].

Assume that f is L-Lipschitz continuous. We prove that,

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty, \tag{37}$$

which will establish the result from theorem 4.

Fix an $h > 0$. Define $g_h : \mathbb{R} \to \mathbb{C}$, by

$$g_h(x) = f(x+h) - f(x-h). \tag{38}$$

Notice that,

$$\begin{aligned}
\widehat{g_h}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g_h(x)e^{-inx}\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x+h)-f(x-h))e^{-inx}\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+h)e^{-inx}\mathrm{d}x - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-h)e^{-inx}\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi+h}^{\pi+h} f(y)e^{-in(y-h)}\mathrm{d}y - \frac{1}{2\pi}\int_{-\pi-h}^{\pi-h} f(y)e^{-in(y+h)}\mathrm{d}y \\
&=_{(a)} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-in(y-h)}\mathrm{d}y - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-in(y+h)}\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)(e^{inh}-e^{-inh})e^{-iny}\mathrm{d}y \\
&= 2i\sin(nh)\hat{f}(n),
\end{aligned} \tag{39}$$

where (a) follows since, $y \mapsto f(y)e^{-iny}$ is 2π-periodic. Hence from Parseval's identity (theorem 2) we have,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|g_h(x)|^2\mathrm{d}x = ||g_h||^2 = \sum_{n=-\infty}^{\infty}|\widehat{g_h}(n)|^2 = \sum_{n=-\infty}^{\infty} 4|\sin(nh)|^2|\hat{f}(n)|^2 \tag{40}$$

But notice that,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|g_h(x)|^2\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h)-f(x-h)|^2\mathrm{d}x \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}(2hL)^2\mathrm{d}x = 4h^2L^2. \tag{41}$$

Hence combining with (40), we have,

$$\sum_{n=-\infty}^{\infty} |\sin(nh)|^2 |\hat{f}(n)|^2 \leq h^2 L^2. \tag{42}$$

Now let $p \in \mathbb{N}$, and let $h = \pi/2^{p+1}$. Notice that for $n \in \mathbb{N}$ such that $2^{p-1} < |n| \leq 2^p$, we have that,

$$\frac{\pi}{4} < |nh| \leq \frac{1}{2}. \tag{43}$$

Hence,

$$\frac{1}{\sqrt{2}} < |\sin(nh)|. \tag{44}$$

Hence from (42), we have,

$$h^2 L^2 \geq \sum_{n=-\infty}^{\infty} |\sin(nh)|^2 |\hat{f}(n)|^2 \geq \sum_{2^{p-1}<|n|\leq 2^p} |\sin(nh)|^2 |\hat{f}(n)|^2 \geq \frac{1}{2} \sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2, \tag{45}$$

which after substituting for h transforms into,

$$\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \leq \frac{L^2\pi^2}{2^{2p+1}} \tag{46}$$

Notice that from Cauchy-Schwartz inequality and the above inequality, we have,

$$\begin{aligned} \left(\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)| \right)^2 &\leq \left(\sum_{2^{p-1}<|n|\leq 2^p} 1 \right) \left(\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \right) = 2^p \left(\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \right) \\ &\leq 2^p \frac{L^2\pi^2}{2^{2p+1}} = \frac{L^2\pi^2}{2^{p+1}}. \end{aligned} \tag{47}$$

Hence we have that,

$$\sum_{2^p<|n|\leq 2^p} |\hat{f}(n)| \leq \frac{\pi L}{2^{(p+1)/2}}. \tag{48}$$

Hence,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |\hat{f}(n)| &= |\hat{f}(0)| + |\hat{f}(1)| + |\hat{f}(-1)| + \sum_{p=1}^{\infty} \sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)| \\ &\leq |\hat{f}(0)| + |\hat{f}(1)| + |\hat{f}(-1)| + \sum_{p=1}^{\infty} \frac{\pi L}{\sqrt{2}^{(p+1)}} \\ &= |\hat{f}(0)| + |\hat{f}(1)| + |\hat{f}(-1)| + \frac{\pi L}{\sqrt{2}-1} < \infty. \end{aligned} \tag{49}$$

Hence we have the desired result. □

As a consequence of the above theorem, we have the following corollary.

Corollary 5.1. *Consider $f \in C^1([-\pi, \pi]; \mathbb{C})$. The $S_N(f)$ converges uniformly to f.*

Although the above establishes the uniform convergence of Fourier series for $C^1([-\pi, \pi]; \mathbb{C})$ functions, more can be said regarding the rate of decay of their Fourier coefficients.

Theorem 6. *The Fourier coefficients satisfy $\hat{f}(n) = \mathcal{O}(1/|n|^m)$ as $|n| \to \infty$, whenever $f \in C^m([-\pi, \pi]; \mathbb{C})$.*

Proof. Let $f^{(m)}$ denote the m-th derivative of f. Notice that applying lemma 2 m-times yields,

$$\widehat{f^{(m)}}(n) = (in)^m \hat{f}(n). \tag{50}$$

Hence,

$$\begin{aligned} |\hat{f}(n)| &= \frac{1}{|n|^m} \left| \widehat{f^{(m)}}(n) \right| = \frac{1}{|n|^m} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{(m)}(y) e^{-iny} \mathrm{d}y \right| \leq \frac{1}{2\pi |n|^m} \int_{-\pi}^{\pi} |f^{(m)}(y)| \mathrm{d}y \\ &= \frac{C}{|n|^m}, \end{aligned} \tag{51}$$

where $C = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f^{(m)}(y)| \mathrm{d}y$ is independent of n. □

Notice that the above theorem, along with theorem 4, establish that the Fourier series of $C^2([-\pi, \pi]; \mathbb{C})$ functions converge uniformly.

B. Point-wise Convergence of the Fourier Series of Differentiable Functions

Consider $f \in L^2_{\mathcal{R}}([-\pi, \pi]; \mathbb{C})$ such that f is differentiable at a point $x \in \mathbb{R}$. Notice that section IV-A does not guarantee the uniform convergence of the Fourier series for this case. Nevertheless, it can be established that $S_N(f)(x)$ converges to $f(x)$. Hence if f is differentiable, we have that $S_N(f)$ converges point-wise to f. It should be noted that for this result to hold, we do not require continuous differentiability. We establish the result using the following theorem.

Theorem 7. *Consider $f \in L^2_{\mathcal{R}}([-\pi, \pi]; \mathbb{C})$. Moreover, assume that f is differentiable at x. Then $S_N(f)(x)$ converges to $f(x)$.*

Proof. This proof follows the proof provided in [2]. We begin by defining the function $F : [-\pi, \pi] \to \mathbb{C}$,

$$F(y) = \begin{cases} \frac{f(x-y)-f(x)}{y} & \text{if } y \neq 0 \\ -f^{'}(x) & y = 0 \end{cases} \tag{52}$$

We first establish that F is Riemann integrable in $[-\pi, \pi]$. We state this as a lemma here and attach the proof in the appendix.

Lemma 7. *The function F defined in* (52) *is Riemann integrable in* $[-\pi, \pi]$.

Proof. See Appendix B □

Now we proceed to the main proof. Notice that,

$$\begin{aligned}
S_N(f)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)D_N(y)\mathrm{d}y - f(x) \\
&=_{(a)} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)D_N(y)\mathrm{d}y - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_N(y)\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x-y)-f(x))D_N(y)\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} yF(y)D_N(y)\mathrm{d}y \\
&=_{(b)} \frac{1}{2\pi}\int_{-\pi}^{\pi} yF(y)\frac{\sin\left(\left(N+\frac{1}{2}\right)y\right)}{\sin\left(\frac{y}{2}\right)}\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} yF(y)\left(\frac{\sin(Ny)\cos\left(\frac{y}{2}\right)+\cos(Ny)\sin\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)}\right)\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{yF(y)\cos\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)}\sin(Ny) + \frac{1}{2\pi}\int_{-\pi}^{\pi} yF(y)\cos(Ny)\,\mathrm{d}y, \qquad (53)
\end{aligned}$$

where (a) follows due to lemma 6-2), and (b) follows due to lemma 6-1).

Since F is Riemann integrable in $[-\pi, \pi]$, and $y \mapsto \frac{y\cos\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)}$ is continuous, we have that, $y \mapsto \frac{yF(y)\cos\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)}$, and $y \mapsto yF(y)$ are Riemann integrable in $[-\pi, \pi]$. Hence from the Riemann–Lebesgue lemma (lemma 3), we have that,

$$\begin{aligned}
\lim_{N\to\infty} S_N(f)(x) - f(x) = \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{yF(y)\cos\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)}\sin(Ny) \\
+ \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} yF(y)\cos(Ny)\,\mathrm{d}y = 0, \qquad (54)
\end{aligned}$$

which establishes our claim. □

C. A Continuous Function with a Diverging Fourier Series

In this section, we prove the existence of a $C([-\pi, \pi]; \mathbb{C})$ function whose Fourier series diverges at a point. Although it is possible to provide an explicit construction of such a function, we will only focus on proving its existence. We first state the Banach-Steinhaus theorem, which will be used in the proof.

Theorem 8. *(Banach-Steinhaus Theorem): Let X and Y be Banach spaces, and let $\{T_\alpha | \alpha \in A\}$ be a collection of elements of $\mathcal{B}(X, Y)$ (bounded linear transformation from X to Y), where A is an index set. Then either,*

$$\sup_{\alpha \in A} ||T_\alpha||_{X \to Y} < \infty, \tag{55}$$

or,

$$\sup_{\alpha} ||T_\alpha x||_X = \infty, \tag{56}$$

for all $x \in G$, where G is a dense set in X.

Proof. See Appendix C. We do not prove the full statement. Instead, we prove that if,

$$\sup_{\alpha \in A} ||T_\alpha||_{X \to Y} = \infty \tag{57}$$

then there exists $x \in G$ for which,

$$\sup_{\alpha} ||T_\alpha x||_X = \infty. \tag{58}$$

This is enough for our purposes. The proof is taken from [3]. □

Now we prove the main result of this section.

Theorem 9. *There exists $f \in C([-\pi, \pi]; \mathbb{C})$, whose Fourier series diverges at 0.*

Proof. The proof is adapted from [4]. Consider the two spaces $X = (C([-\pi, \pi]; \mathbb{C}), ||.||_u)$, and $Y = (\mathbb{C}, |.|)$, Notice that X and Y are Banach spaces. Also define the sequence $\{T_n\}_{n=1}^{\infty}$ of linear transformations from X to Y where T_n is defined by,

$$T_n f = S_N(f)(0). \tag{59}$$

Notice that the above is a linear transformation due to lemma 5. Moreover for any $f \in C([-\pi, \pi]; \mathbb{C})$,

$$|S_N(f)(0)| \leq ||S_N(f)||_u = ||D_N * f||_u \leq_{(a)} ||D_N||_1 ||f||_u, \tag{60}$$

where (a) follows from lemma 4-4). Hence, we have that $T_n \in \mathcal{B}(X, y)$, and $||T_N||_{X \to Y} \leq ||D_N||_1$.

Now we prove that $||T_N||_{X\to Y} = ||D_N||_1$. We do this as follows. For any $\varepsilon > 0$, we construct a $\tilde{g}_\varepsilon \in C([-\pi,\pi];\mathbb{C})$, such that, $|S_N(\tilde{g}_\varepsilon)(0)| > ||D_N||_1 - \varepsilon$, and $||\tilde{g}_\varepsilon||_u = 1$, which will establish the result.

First, consider the function $g \in B([-\pi,\pi];\mathbb{C})$ given by,

$$g(x) = \begin{cases} 1 & \text{if } D_N(x) > 0 \\ 0 & \text{if } D_N(x) = 0 \\ -1 & \text{if } D_N(x) < 0 \end{cases} \tag{61}$$

Notice that $||g||_1 = ||g||_u = 1$. Also notice that $D_N(x)g(x) = |D_N(x)|$ for each x. Since D_N is a continuous function, the set of discontinuities $\mathcal{G}$ of g is the set points at which D_N is zero. Notice that from lemma 6-1), there are only finitely many such points in $[-\pi,\pi]$. Let us denote $\mathcal{G}\cap[-\pi,\pi] = \{a_i\}_{i=1}^M$ for some $M \in \mathbb{N}$, where $a_j < a_{j+1}$ for all $1 \le j < M$. Let us also define $a_0 = -\pi$ and $a_{M+1} = \pi$ (Notice that $\pi, -\pi \notin \mathcal{G}$).

We use g, in order to construct a $\tilde{g}_\varepsilon$. Let $\delta > 0$ be such that,

$$\delta < \frac{\varepsilon\pi}{2M||D_N||_u} \tag{62}$$

and

$$\delta < \frac{a_j - a_{j-1}}{2}, \tag{63}$$

for all $1 \le j \le M+1$. We obtain $\tilde{g}_\varepsilon$ in $[-\pi,\pi]$ by augmenting g in the intervals $(a_k - \delta, a_k + \delta)$ for each $1 \le k \le M$. Notice that (63) guarantees that the intervals in $\{(a_k-\delta, a_k+\delta)\}_{k=1}^M$ do not intersect. For each k such that $1 \le k \le M$, $\tilde{g}_\varepsilon|_{(a_k-\delta,a_k+\delta)}$ is defined to be the straight line joining $g(a_k-\delta)$ and $g(a_k+\delta)$, which makes $\tilde{g}_\varepsilon$ is continuous. Moreover, due to (63), $||\tilde{g}_\varepsilon||_u = 1$. Also,

$$\begin{aligned}
|S_N(\tilde{g}_\varepsilon)(0)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} D_N(-t)\tilde{g}_\varepsilon(t)\mathrm{d}t\right| =_{(a)} \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} D_N(t)\tilde{g}_\varepsilon(t)\mathrm{d}t\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} D_N(t)(\tilde{g}_\varepsilon(t) - g(t) + g(t))\mathrm{d}t\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} D_N(t)(\tilde{g}_\varepsilon(t) - g(t))\mathrm{d}t + \int_{-\pi}^{\pi} D_N(t)g(t)\mathrm{d}t\right| \\
&\ge_{(b)} \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} D_N(t)g(t)\mathrm{d}t\right| - \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} D_N(t)(\tilde{g}_\varepsilon(t) - g(t))\mathrm{d}t\right| \\
&=_{(c)} \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} D_N(t)g(t)\mathrm{d}t\right| - \frac{1}{2\pi}\left|\sum_{k=1}^{M}\int_{a_k-\delta}^{a_k+\delta} D_N(t)(\tilde{g}_\varepsilon(t) - g(t))\mathrm{d}t\right|
\end{aligned}$$

$$= ||D_N||_1 - \frac{1}{2\pi}\left|\sum_{k=1}^{M}\int_{a_k-\delta}^{a_k+\delta} D_N(t)(\tilde{g}_\varepsilon(t) - g(t))\mathrm{d}t\right|$$
$$\geq ||D_N||_1 - \frac{1}{2\pi}\sum_{k=1}^{M}\int_{a_k-\delta}^{a_k+\delta} |D_N(t)||\tilde{g}_\varepsilon(t) - g(t)|\mathrm{d}t$$
$$\geq_{(d)} ||D_N||_1 - \frac{1}{2\pi}\sum_{k=1}^{M}\int_{a_k-\delta}^{a_k+\delta} 2||D_N||_u\mathrm{d}t$$
$$= ||D_N||_1 - \frac{2M\delta||D_N||_u}{\pi} > ||D_N||_1 - \varepsilon, \tag{64}$$

where (a) follows due to (6)-1), for (b) we use triangle inequality, for (c), we have used that g and $\tilde{g}_\varepsilon$ agree on $[-\pi,\pi]$ everywhere except in $\cup_{k=1}^{M}(a_k-\delta, a_k+\delta)$, and for (d) we have used that $|\tilde{g}_\varepsilon(t) - g(t)| \leq 2$ for all t. Hence we are done.

Now notice that from lemma 6-3), we have that,

$$\sup_{n\in\mathbb{N}} ||D_n||_1 = \infty, \tag{65}$$

since $\sum_{k=2}^{\infty} = \infty$. Since $||D_n||_1 = ||T_n||_{X\to Y}$ for each $n \in \mathbb{N}$, we have,

$$\sup_{n\in\mathbb{N}} ||T_n||_{X\to Y} = \infty. \tag{66}$$

Hence from the Banach-Steinhaus theorem, there exists $f \in C([-\pi,\pi];\mathbb{C})$, such that,

$$\sup_{n\in N} ||T_n f|| = \sup_{n\in N} ||S_n(f)(0)|| = \infty. \tag{67}$$

In other words, the Fourier series of f diverges at 0. Hence we have the claim. □

V. Césaro summability of Fourier Series

Although the Fourier series of a general continuous function may not necessarily converge point-wise to the function according to section IV-C, it can be established that if $f \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$ is continuous at x, then $(S_n(f)(x))_{n=1}^{\infty}$ is Césaro summable to $f(x)$. In order to prove this, we first introduce Fejer Kernels.

The Fejer kernel F_N is defined by,

$$F_N = \frac{1}{N}\sum_{n=0}^{N-1} D_n. \tag{68}$$

It can be proved that $\{F_N\}_{N=1}^{\infty}$ is an approximate identity. We will prove this in a while. First, we will look at the implication of this statement.

Theorem 10. *For a function $f \in L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C})$, continuous at a point $x \in [-\pi,\pi]$, the Fourier series of f is Césaro summable to $f(x)$ at x. In particular, if $f \in C([-\pi,\pi];\mathbb{C})$, since $f*F_n \to f$ uniformly, the Fourier series of f is Césaro summable to f.*

Proof. If f is continuous at $x \in [-\pi,\pi]$, then $(f*F_n)(x) \to f(x)$ from theorem 3. But notice that,

$$(f*F_n) = f*\left(\frac{1}{N}\sum_{n=0}^{N-1} D_n\right) =_{(a)} \frac{1}{N}\sum_{n=0}^{N-1}(f*D_n) = \frac{1}{N}\sum_{n=0}^{N-1} S_n(f), \tag{69}$$

where equality (a) follows from lemma 4-3). This means that, $\frac{1}{N}\sum_{n=0}^{N-1} S_n(f)(x) \to f(x)$. In other words, the Fourier series of f is Césaro summable to $f(x)$ at x. □

Now we prove that, $\{F_n\}_{n=1}^{\infty}$ is infact an approximate identity.

Theorem 11. $\{F_n\}_{n=1}^{\infty}$*, where F_n is defined in* (68) *is an approximate identity.*

Proof. First, we prove that

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}, \tag{70}$$

for all $x \in \mathbb{R}$. We proceed by induction. Notice that,

$$F_1(x) = D_0(x) =_{(a)} \frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} = 1 \tag{71}$$

where (a) follows from lemma 6-1). Hence the result is true for $N=1$. Now assume that the result is true for $N=k$, where $k \geq 1$. Hence,

$$F_k(x) = \frac{1}{k}\frac{\sin^2(kx/2)}{\sin^2(x/2)} \tag{72}$$

Hence,

$$\begin{aligned}
F_{k+1}(x) &= \frac{kF_k(x)+D_k(x)}{(k+1)} =_{(a)} \frac{1}{k+1}\left(\frac{\sin^2\left(\frac{kx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)} + \frac{\sin\left(\left(k+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}\right) \\
&= \frac{1}{k+1}\left(\frac{\sin^2\left(\frac{kx}{2}\right)+\sin\left(\frac{x}{2}\right)\sin\left(\left(k+\frac{1}{2}\right)x\right)}{\sin^2\left(\frac{x}{2}\right)}\right) \\
&=_{(b)} \frac{1}{2(k+1)}\left(\frac{1-\cos(kx)+\cos(kx)-\cos((k+1)x)}{\sin^2\left(\frac{x}{2}\right)}\right) \\
&= \frac{1}{2(k+1)}\left(\frac{1-\cos((k+1)x)}{\sin^2\left(\frac{x}{2}\right)}\right) =_{(c)} \frac{1}{(k+1)}\frac{\sin^2\left(\frac{(k+1)x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}
\end{aligned} \tag{73}$$

where (a) follows from the induction hypothesis and lemma 6-1), and for (b) and (c), we have used $2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$. This completes the induction, and we have the result.

Now we prove that $\{F_n\}_{n=1}^{\infty}$ satisfies the three properties of an approximate identity.

1) Notice that,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y)\mathrm{d}y = \frac{1}{2\pi N}\sum_{n=0}^{N-1}\int_{-\pi}^{\pi} D_n(y)\mathrm{d}y =_{(a)} 1 \tag{74}$$

where (a) follows from lemma 6-2)

2) This follows from 1) since F_N is non-negative from (70).

3) Notice that from (70) we have that F_N is an even function. Hence we need to establish that,

$$\lim_{N\to\infty}\int_{\delta}^{\pi} |F_N(x)|\mathrm{d}x = 0, \tag{75}$$

for any $\delta \in (0,\pi)$. But notice that $x \mapsto \sin^2(x/2)$ is a non-decreasing function in $[0,\pi]$. Hence $\sin^2(x/2) \geq \sin^2(\delta/2) = c > 0$ for all $x \in [\delta,\pi]$. Hence,

$$\begin{aligned}
\lim_{N\to\infty}\int_{\delta}^{\pi} |F_N(x)|\mathrm{d}x &= \lim_{N\to\infty}\frac{1}{N}\int_{\delta}^{\pi}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}\mathrm{d}x \leq \lim_{N\to\infty}\frac{1}{Nc}\int_{\delta}^{\pi}\sin^2(Nx/2)\mathrm{d}x \\
&= \lim_{N\to\infty}\frac{1}{2Nc}\int_{\delta}^{\pi}(1-\cos(Nx))\mathrm{d}x \\
&= \lim_{N\to\infty}\frac{1}{2Nc}\left[x - \frac{1}{N}\sin(Nx)\right]_{\delta}^{\pi} \\
&= \lim_{N\to\infty}\frac{1}{2Nc}\left(\pi - \delta + \frac{1}{N}\sin(N\delta)\right) \\
&\leq \lim_{N\to\infty}\frac{1}{2Nc}\left(\pi - \delta + \frac{1}{N}\right) = 0
\end{aligned} \tag{76}$$

□

VI. Conclusions

In this article, we established that the Fourier series of a Lipschitz continuous function converges uniformly to the function. We also established the point-wise convergence of the Fourier series at the points at which the function is differentiable. We also proved the existence of a continuous function whose Fourier series diverges at a point. Finally, we established the Césaro summability of the Fourier series at the points of continuity of the function.

APPENDIX A
PROOF OF LEMMA 6

We will prove each statement separately.

1) Notice that,

$$\begin{aligned} D_N(y) &= \sum_{n=-N}^{N} e^{iny} = e^{-iNy}\sum_{n=0}^{2N} e^{iny} =_{(a)} e^{-iNy}\left(\frac{1-e^{(2N+1)iy}}{1-e^{iy}}\right) \\ &= \frac{e^{-i\left(N+\frac{1}{2}\right)y}}{e^{-\frac{iy}{2}}}\left(\frac{1-e^{(2N+1)iy}}{1-e^{iy}}\right) = \frac{e^{-i\left(N+\frac{1}{2}\right)y} - e^{i\left(N+\frac{1}{2}\right)y}}{e^{-\frac{iy}{2}} - e^{\frac{iy}{2}}} \\ &= \frac{\sin\left(\left(N+\frac{1}{2}\right)t\right)}{\sin\left(\frac{t}{2}\right)}, \end{aligned} \tag{77}$$

where (a) follows from the formulae for the sum of the first $2N+1$ terms of a geometric series.

2) Notice that,

$$\begin{aligned} \int_{-\pi}^{\pi} D_N(y)\mathrm{d}y &= \int_{-\pi}^{\pi}\sum_{n=-N}^{N} e^{iny}\mathrm{d}y = \sum_{n=-N}^{N}\int_{-\pi}^{\pi} e^{iny}\mathrm{d}y \\ &= \sum_{\substack{n=-N \\ n\neq 0}}^{N}\left[\frac{e^{iny}}{in}\right]_{-\pi}^{\pi} + 2\pi = 2\pi \end{aligned} \tag{78}$$

3) This proof follows the outline provided Chapter-2, Problem-2 in [2]. First, notice that, from part 2),

$$|D_N(y)| = \frac{\left|\sin\left(\left(N+\frac{1}{2}\right)y\right)\right|}{\left|\sin\left(\frac{y}{2}\right)\right|} \geq 2\frac{\left|\sin\left(\left(N+\frac{1}{2}\right)y\right)\right|}{|y|}, \tag{79}$$

where we have used the inequality $|\sin(x)| \leq |x|$. Hence, notice,

$$\begin{aligned} ||D_N||_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(y)|\mathrm{d}y \geq \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\left|\sin\left(\left(N+\frac{1}{2}\right)y\right)\right|}{|y|}\mathrm{d}y \\ &=_{(a)} \frac{1}{\pi}\int_{-\pi\left(N+\frac{1}{2}\right)}^{\pi\left(N+\frac{1}{2}\right)}\frac{|\sin(\theta)|}{|\theta|}\mathrm{d}\theta \\ &=_{(b)} \frac{2}{\pi}\int_{0}^{\pi\left(N+\frac{1}{2}\right)}\frac{|\sin(\theta)|}{|\theta|}\mathrm{d}\theta \\ &\geq \frac{2}{\pi}\int_{\pi}^{N\pi}\frac{|\sin(\theta)|}{|\theta|}\mathrm{d}\theta \\ &= \sum_{k=1}^{N-1}\frac{2}{\pi}\int_{k\pi}^{(k+1)\pi}\frac{|\sin(\theta)|}{|\theta|}\mathrm{d}\theta \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{k=1}^{N-1} \frac{2}{\pi} \int_{k\pi}^{(k+1)\pi} \frac{|\sin(\theta)|}{(k+1)\pi} \mathrm{d}\theta \\
&= \frac{2}{\pi^2} \sum_{k=1}^{N-1} \frac{1}{(k+1)} \int_{k\pi}^{(k+1)\pi} |\sin(\theta)| \, \mathrm{d}\theta \\
&=_{(c)} \frac{2}{\pi^2} \sum_{k=1}^{N-1} \frac{1}{(k+1)} \left| \int_{k\pi}^{(k+1)\pi} \sin(\theta) \, \mathrm{d}\theta \right| \\
&= \frac{2}{\pi^2} \sum_{k=1}^{N-1} \frac{1}{(k+1)} \left| [\cos(\theta)]_{k\pi}^{(k+1)\pi} \right| \\
&= \frac{4}{\pi^2} \sum_{k=2}^{N} \frac{1}{k} \qquad (80)
\end{aligned}$$

where in (a), we have used the change of variables $\theta = \left(N + \frac{1}{2}\right) y$, for (b), we have used the fact that $|\sin(\theta)|/|\theta|$ is an even function, and in (c), we have used the fact that $\sin$ function does not switch signs in $(k\pi, (k+1)\pi)$ for all $k \in \mathbb{N}$.

APPENDIX B
PROOF OF LEMMA 7

For this, first, we prove that F is bounded. Notice that since f is Riemann integrable in $[-\pi, \pi]$, it is also bounded in $[-\pi, \pi]$. Let $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Also since f is differentiable at x, there exists, $\delta > 0$ such that, $y \in (-\delta, \delta)$ implies,

$$1 > \left| \frac{f(x-y) - f(x)}{-y} - f^{'}(x) \right| = |F(y) + f^{'}(x)|, \tag{81}$$

which implies that, $|y| < \delta$, implies,

$$|F(y)| < |f^{'}(x)| + 1, \tag{82}$$

Also, if $|y| > \delta$, we have that,

$$|F(y)| = \left| \frac{f(x-y) - f(x)}{y} \right| < \frac{1}{\delta}(|f(x-y)| + |f(x)|) < \frac{2M}{\delta}. \tag{83}$$

Hence in general, we have,

$$|F(y)| \leq \max \left\{ |f^{'}(x)| + 1, \frac{2M}{\delta} \right\} = B, \tag{84}$$

which implies that F is bounded.

Now we prove that F is Riemann integrable in $[-\pi, \pi]$. This is a direct consequence of the Riemann-Lesbague theorem proved in [1] since the set of discontinuities of F is a subset of the set of discontinuities of f along with 0. We will also provide the following proof, which does not use the Riemann-Lesbague theorem. Notice that, F is Riemann integrable in $[-\pi, -\delta]$ and $[\delta, \pi]$ for any $\delta > 0$ (This follows since both f and $\frac{1}{y}$ are Riemann integrable in the considered intervals). Pick any $\varepsilon > 0$. Choose partitions P_1 and P_2 of $[-\pi, -\varepsilon/(12B)]$, and $[\varepsilon/(12B), \pi]$ respectively such that, $|U(P_1, F) - L(P_1, F)| < \varepsilon/3$ and $|U(P_2, F) - L(P_2, F)| < \varepsilon/3$. Now consider the partition defined by $P = P_1 \cup P_2$. Notice that,

$$U(P, F) \leq U(P_1, F) + U(P_2, F) + 2B\frac{\varepsilon}{12B}, \tag{85}$$

where the last term is due to the interval $[-\varepsilon/(12B), \varepsilon/(12B)]$ in P. Similarly,

$$L(P, F) \geq L(P_1, F) + L(P_2, F) - 2B\frac{\varepsilon}{12B}. \tag{86}$$

Hence,

$$U(P, F) - L(P, F) \leq U(P_1, F) - L(P_1, F) + U(P_2, F) - L(P_2, F) + \frac{\varepsilon}{3} < \varepsilon. \tag{87}$$

Hence, F is Riemann integrable in $[-\pi, \pi]$ as desired.

Appendix C
Proof of Banach Steinhaus Theorem

We start with the following lemma.

Lemma 8. *Consider any $T \in \mathcal{B}(X, Y)$. Then for any $x \in X$ and $r > 0$, we have,*

$$\sup_{y \in B_X(x,r)} ||Ty||_Y \geq r||T||_{X \to Y} \tag{88}$$

.

Proof. Consider $z \in B_X(0, r)$. Notice that,

$$\max\{||T(x+z)||_Y, ||T(x-z)||_Y\} \geq \frac{1}{2}(||T(x+z)||_Y + ||T(x-z)||_Y) \geq ||Tz||_Y, \tag{89}$$

where we have used the triangle inequality for the last step. Taking the supremum, we have that,

$$\sup_{z \in B_X(0,r)} \max\{||T(x+z)||_Y, ||T(x-z)||_Y\} \geq \sup_{z \in B_X(0,r)} ||Tz||_Y. \tag{90}$$

But notice that, $\sup_{z \in B_X(0,r)} ||Tz||_Y = r||T||_{X \to Y}$, from the definition of $||T||_{X \to Y}$. Also,

$$\sup_{z \in B_X(0,r)} \max\{||T(x+z)||_Y, ||T(x-z)||_Y\} = \sup_{z \in B_X(0,r)} ||T(x+z)||_Y = \sup_{y \in B_X(x,r)} ||Ty||_Y. \tag{91}$$

Hence we have the desired result. □

Now assume that $\sup_\alpha ||T_\alpha||_{X\to Y} = \infty$*. Choose a sequence,* $(T_n)_{n=1}^\infty$ *of* $\{T_\alpha|\alpha \in A\}$ *such that* $||T_n||_{X\to Y} \geq 4^n$*. Now we construct a sequence* $(x_n)_{n=0}^\infty$ *of elements in* X *inductively as follows.*

Start with $x_0 = 0$*. Assume we have constructed* $(x_n)_{n=0}^{k-1}$*, where* $k \geq 1$*. Notice that, from lemma 8 and the definition of supremum, there exists* $y \in B(x_{k-1}, 3^{-k})$ *such that*

$$||T_k y||_Y \geq \frac{2}{3}3^{-k}||T_k||_{X\to Y}. \tag{92}$$

We set $x_k = y$*. Notice that,* $(x_n)_{n=0}^\infty$ *constructed this way is a Cauchy sequence since,* $||x_k - x_{k-1}||_X < 3^{-k}$*. Since* X *Banach space, there exists* $x \in X$ *such that* $\lim_{n\to\infty} ||x_n - x||_X = 0$*. We now prove that,*

$$||x_k - x||_X \leq \frac{1}{2}3^{-k}. \tag{93}$$

for all $k \in \mathbb{N}$*. For proof, pick an arbitrary* $k \in \mathbb{N}$*. We prove the result for this* k*. Set* $\varepsilon > 0$ *such that* $\varepsilon < \frac{1}{2}3^{-k}$*. Notice that, there exists* $N \in \mathbb{N}$ *such that* $n \geq N$ *implies,*

$$||x_n - x||_X < \varepsilon. \tag{94}$$

If $k \geq N$*, we are done. Hence assume* $k < N$*. Notice that,*

$$\begin{aligned} ||x_k - x_N||_X &= \left\|\sum_{j=k}^{N-1}(x_j - x_{j+1})\right\| \leq_{(a)} \sum_{j=k}^{N-1} \|x_j - x_{j+1}\| \\ &\leq_{(b)} \sum_{j=k}^{N-1} 3^{-(j+1)} \leq \sum_{j=k}^{\infty} 3^{-(j+1)} = \frac{3^{-(k+1)}}{1-(1/3)} = \frac{1}{2}3^{-k} \end{aligned} \tag{95}$$

where (a) follows from the triangle inequality, and (b) follows from (93)*. Hence,*

$$||x_k - x||_X = ||x_k - x_N + x_N - x||_X \leq ||x_k - x_N||_X + ||x_N - x||_X \leq \frac{1}{2}3^{-k} + \varepsilon. \tag{96}$$

But notice that our choice of ε *is arbitrary. Hence, we have,*

$$||x_k - x||_X \leq \frac{1}{2}3^{-k}, \tag{97}$$

as desired.

Now notice that,

$$\begin{aligned} ||T_n x||_Y &= ||T_n(x - x_n + x_n)||_Y \geq_{(a)} ||T_n x_n||_Y - ||T_n(x - x_n)||_Y \\ &\geq_{(b)} \frac{2}{3}3^{-n}||T_n||_{X\to Y} - ||x - x_n||_x ||T_n||_{X\to Y} \\ &\geq \frac{2}{3}3^{-n}||T_n||_{X\to Y} - \frac{1}{2}3^{-n}||T_n||_{X\to Y} \geq \frac{1}{6}3^{-n}||T_n||_{X\to Y} \geq \frac{1}{6}\left(\frac{4}{3}\right)^n, \end{aligned} \tag{98}$$

where (a) follows from the triangle inequality, $||T_n x_n||_Y \geq \frac{2}{3}3^{-n}||T_n||_{X\to Y}$ in (b) follows from the construction of x_n, and the last inequality follows from the choice of the sequence $(T_n)_{n=1}^{\infty}$. Hence,

$$\sup_{\alpha\in A}||T_\alpha x||_Y \geq \sup_{n\in\mathbb{N}}||T_n x||_Y \geq \sup_{n\in\mathbb{N}}\frac{1}{6}\left(\frac{4}{3}\right)^n = \infty, \tag{99}$$

as desired.

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