Sub-Gaussian Random Variables, Tail Probability Bounds With Random Number of Samples

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I. Introduction

Bounding the tail probability of an average of a random number of samples of a collection of random variables has application in multi-armed bandit problems [1]. In this article, we will be summarizing a few results on this.

II. Sub-Gaussian Random Variables

A zero-mean random variable X is σ-sub-Gaussian if,

$$\mathbb{E}\{e^{tX}\} \leq e^{\frac{\sigma^2 t^2}{2}}, \tag{1}$$

for all $t \in \mathbb{R}$. It can be shown that a zero-mean Gaussian random variable with standard deviation σ is σ-sub-Gaussian. We have the following lemma, which will be useful.

Lemma 1: For a zero-mean σ-sub-Gaussian random variable X, we have that, Var$(X) \leq \sigma$.

Proof: This can be proved using the Taylor expansion and the dominated convergence theorem. We will omit the proof for brevity. ■

Lemma 2: If X_1, and X_2 are zero-mean independent σ_1-sub-Gaussian, and σ_2-sub-Gaussian random variables, respectively, then $X_1 + X_2$ is $\sqrt{\sigma_1^2 + \sigma_2^2}$-sub-Gaussian.

Proof: Notice that for any $t \in \mathbb{R}$,

$$\mathbb{E}\{e^{t(X_1+X_2)}\} = \mathbb{E}\{e^{tX_1}e^{tX_2}\} =_{(a)} \mathbb{E}\{e^{tX_1}\}\mathbb{E}\{e^{tX_2}\} \leq e^{\frac{\sigma_1^2 t^2}{2}} e^{\frac{\sigma_2^2 t^2}{2}} = e^{\frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}, \tag{2}$$

where (a) follows from the independence of X_1, and X_2. Hence, we are done. ■

III. Chernoff Bound for Sub-Gaussian Random variables

The following lemma establishes the Chernoff bound on the right tail probability for σ-sub-Gaussian Random variables.

Lemma 3 (Chernoff bounds): Given a zero-mean σ-sub-Gaussian random variable X and $\varepsilon > 0$. Then, we have that,

$$P\left(X \geq \varepsilon\right) \leq e^{-\frac{\varepsilon^2}{2\sigma^2}}. \tag{3}$$

Also substituting $\varepsilon = \sqrt{2\sigma^2 \log(1/\delta)}$ in the above, we have that,

$$P\left(X \geq \sqrt{2\sigma^2 \log\left(\frac{1}{\delta}\right)}\right) \leq \delta. \tag{4}$$

Proof: Notice that for any $t > 0$, we have that,

$$\begin{aligned} P(X \geq \varepsilon) = P(tX \geq t\varepsilon) = P(e^{tX} \geq e^{t\varepsilon}) \leq_{(a)} \frac{\mathbb{E}\{e^{tX}\}}{e^{t\varepsilon}} \leq_{(b)} e^{\frac{\sigma^2 t^2}{2} - t\varepsilon} \\ \leq e^{\frac{\sigma^2}{2}\left(t - \frac{\varepsilon}{\sigma^2}\right)^2 - \frac{\varepsilon^2}{2\sigma^2}} \leq e^{-\frac{\varepsilon^2}{2\sigma^2}} \end{aligned} \tag{5}$$

where (a) follows from the Markov inequality, and (b) follows since X is σ-sub-Gaussian. ■

The left-tail probability can be bounded using a similar approach. In general, it holds that for a zero-mean σ-sub-Gaussian random variable X,

$$P\left(|X| \geq \varepsilon\right) \leq 2e^{-\frac{\varepsilon^2}{2\sigma^2}}. \tag{6}$$

IV. Tail Probability Bounds on the Average of a Random Number of sub-Gaussian Random Variables

Consider a sequence of independent and identically distributed zero-mean 1-sub-Gaussian random variables, $\{X_t\}_{t=1}^{\infty}$, and a random variable T which takes on positive integer values. Define,

$$\bar{X} = \frac{1}{T}\left(\sum_{t=1}^{T} X_t\right). \tag{7}$$

The problem of bounding the tail probabilities of $\bar{X}$ naturally arises in the multi-armed bandit problem [1], specifically for obtaining worst-case regret bounds for different bandit algorithms. We will look at certain scenarios. First, we will consider the scenario in which T is independent of the collection $\{X_t\}_{t=1}^{\infty}$. In this case, a similar bound to the one derived in Section III holds. When

$\{X_t\}_{t=1}^{\infty}$ is allowed to depend on T the above analysis does not hold in general. Unfortunately, this is the scenario that arises in most practical applications, such as bandit problems. On the bright side, a slightly weaker bound can be derived for the general case. In fact, this weaker bound is still very powerful since it does not affect the asymptotic worst-case regret bounds of bandit algorithms in most cases. For the following section, we follow an exercise in [1].

For convenience, we will define the collection of random variables, $\{X(t)\}_{t=1}^{\infty}$ where,

$$X(t) = \frac{1}{t}\left(\sum_{\tau=1}^{t} X_\tau\right) \tag{8}$$

for all $t \in \mathbb{N}$.

A. $\{X_t\}_{t=1}^{\infty}$ and T are independent

In this case, we have the following lemma. This bound is similar to the classical Chernoff bound obtained in Section III.

Lemma 4: When $\{X_t\}_{t=1}^{\infty}$ and T are independent, we have that for any $\delta \in (0,1)$,

$$P\left(\bar{X} \geq \sqrt{\frac{2\log\frac{1}{\delta}}{T}}\right) \leq \delta. \tag{9}$$

Proof: Notice that,

$$\begin{aligned}
P\left(\bar{X} \geq \sqrt{\frac{2\log\frac{1}{\delta}}{T}}\right) &= \sum_{t=1}^{\infty} P\left(\bar{X} \geq \sqrt{\frac{2\log\frac{1}{\delta}}{T}}\middle| T=t\right) P(T=t) \\
&= \sum_{t=1}^{\infty} P\left(X(t) \geq \sqrt{\frac{2\log\frac{1}{\delta}}{t}}\middle| T=t\right) P(T=t) \\
&=_{(a)} \sum_{t=1}^{\infty} P\left(tX(t) \geq \sqrt{2t\log\frac{1}{\delta}}\right) P(T=t) \\
&\leq_{(b)} \sum_{t=1}^{\infty} \left(e^{-\frac{\left(\sqrt{2t\log\frac{1}{\delta}}\right)^2}{2t}} P(T=t)\right) \\
&= \sum_{t=1}^{\infty} \delta P(T=t) = \delta,
\end{aligned} \tag{10}$$

where (a) follows since $\{X_t\}_{t=1}^{\infty}$ and T are independent, and (b) follows from Lemma 3, since $tX(t)$ is $\sqrt{t}$-sub-Gaussian from Lemma 2 ■

B. $\{X_t\}_{t=1}^{\infty}$ and T are dependent

Unfortunately, in this case, the bound derived in Section IV-A for the independent case does not hold in general. Before moving on to an example, we introduce the law of iterated logarithm.

Lemma 5 (Law of iterated logarithm): Given a collection of zero-mean random variables $\{Y_t\}_{t=1}^{\infty}$ with unit variance. Then we have,

$$\limsup_{t\to\infty} \frac{\sum_{\tau=1}^{t} Y_\tau}{\sqrt{2t\log(\log(t))}} = 1 \text{ (almost surely).} \tag{11}$$

Proof: A proof can be found in [2]. ■

Now, we use the law of iterated logarithm to establish the following lemma, which shows that the bound derived in Section IV-A is not generally valid.

Lemma 6: For any $\delta \in (0,1)$, there exists a random variable T such that,

$$P\left(\bar{X} \geq \sqrt{\frac{2\log\frac{1}{\delta}}{T}}\right) = 1. \tag{12}$$

Proof: Fix $\delta \in (0,1)$. Let γ be the variance of X_t. Notice that from Lemma 1, $\gamma \leq 1$. We define T to be the least positive integer (if it exists) such that

$$TX(T) \geq \sqrt{2T\log\frac{1}{\delta}}. \tag{13}$$

Otherwise, we define $T = 1$. Let us define the two events,

$$A = \left\{X(1) \geq \sqrt{2\log\frac{1}{\delta}}\right\}, \tag{14}$$

and

$$B = \left\{\tau X(\tau) < \sqrt{2\tau\log\frac{1}{\delta}} \;\; \forall \tau \in \mathbb{N}\right\}. \tag{15}$$

Notice that,

$$\{T = 1\} \equiv \{A \cup B\}. \tag{16}$$

We first establish that $P(B) = 0$. Notice that since $\{X_t/\sqrt{\gamma}\}_{t=1}^{\infty}$ is a collection of i.i.d zero mean unit variance random variables, from the law of iterated logarithm we have that

$$P\left(\frac{\sum_{t=1}^{\tau} X_t}{\sqrt{2\tau\gamma\log(\log(\tau))}} \leq \frac{1}{2} \;\; \forall \tau \text{ sufficiently large}\right) = 0. \tag{17}$$

Hence, we have that,

$$P\left(\tau X(\tau) \leq \sqrt{2\tau\left(\frac{\gamma}{4}\right)\log(\log(\tau))}\;\; \forall \tau \text{ sufficiently large}\right) = 0. \tag{18}$$

But notice that,

$$B \subset \left\{\tau X(\tau) \leq \sqrt{2\tau\left(\frac{\gamma}{4}\right)\log(\log(\tau))}\;\; \forall \tau \text{ sufficiently large}\right\} \tag{19}$$

since for any $\omega \in B$ and τ large enough such that,

$$\left(\frac{\gamma}{4}\right)\log(\log(\tau)) \geq \log\left(\frac{1}{\delta}\right) \tag{20}$$

we have that,

$$\tau X(\tau)(\omega) \leq \sqrt{2\tau\left(\frac{\gamma}{4}\right)\log(\log(\tau))}. \tag{21}$$

Hence $P(B) = 0$. Now, we move on to the main proof. Notice that,

$$\begin{aligned}
&P\left(\bar{X} \geq \sqrt{\frac{2\log\frac{1}{\delta}}{T}}\right) = P\left(TX(T) \geq \sqrt{2T\log\frac{1}{\delta}}\right)\\
&= P\left(TX(T) \geq \sqrt{2T\log\frac{1}{\delta}}\middle| T=1\right)P(T=1) + P\left(TX(T) \geq \sqrt{2T\log\frac{1}{\delta}}\middle| T>1\right)P(T>1)\\
&=_{(a)} P\left(X(1) \geq \sqrt{2\log\frac{1}{\delta}}, T=1\right) + P(T>1)\\
&= P\left(A \cap (A \cup B)\right) + P(T>1)\\
&= P(A) + P(A\cup B) - P(A \cup A \cup B) + P(T>1)\\
&= P(A) - P(A\cup B) + P(A\cup B) + P(T>1)\\
&= -P(B) + P(A\cap B) + P(A\cap B) + P(T>1)\\
&\geq P(A\cap B) + P(T>1) = P(T=1) + P(T>1) = 1
\end{aligned} \tag{22}$$

where (a) follows since from the definition of T, we have that if $T > 1$, then,

$$TX(T) \geq \sqrt{2T\log\frac{1}{\delta}}, \tag{23}$$

and the last inequality follows since $P(B) = 0$. ■

Fortunately, a slightly weaker bound can be obtained. The following lemma summarizes the result.

Lemma 7: For any $\delta \in (0,1)$, we have that,

$$P\left(\bar{X} \geq \sqrt{\frac{2\log\frac{T(T+1)}{\delta}}{T}}\right) \leq \delta. \tag{24}$$

Proof: Notice that,

$$\begin{aligned}
P\left(\bar{X} \geq \sqrt{\frac{2\log\frac{T(T+1)}{\delta}}{T}}\right) &= \sum_{t=1}^{\infty} P\left(\bar{X} \geq \sqrt{\frac{2\log\frac{T(T+1)}{\delta}}{T}}, T=t\right) \\
&= \sum_{t=1}^{\infty} P\left(X(t) \geq \sqrt{\frac{2\log\frac{t(t+1)}{\delta}}{t}}, T=t\right) \\
&\leq_{(a)} \sum_{t=1}^{\infty} P\left(X(t) \geq \sqrt{\frac{2\log\frac{t(t+1)}{\delta}}{t}}\right) \\
&= \sum_{t=1}^{\infty} P\left(tX(t) \geq \sqrt{2t\log\frac{t(t+1)}{\delta}}\right) \\
&\leq_{(b)} \sum_{t=1}^{\infty} e^{-\frac{\left(\sqrt{2t\log\frac{t(t+1)}{\delta}}\right)^2}{2t}} \\
&= \sum_{t=1}^{\infty} \frac{\delta}{t(t+1)} = \sum_{t=1}^{\infty}\left(\frac{\delta}{t} - \frac{\delta}{t+1}\right) = \delta
\end{aligned} \tag{25}$$

where (a) follows since for any two events A, B, $P(A,B) \leq P(A)$, and (b) follows from Lemma 3, since $tX(t)$ is $\sqrt{t}$-sub-Gaussian from Lemma 2. ■

REFERENCES

[1] T. Lattimore and C. Szepesvári, *Bandit Algorithms*. Cambridge University Press, 2020.

[2] A. de Acosta, "A New Proof of the Hartman-Wintner Law of the Iterated Logarithm," *The Annals of Probability*, vol. 11, no. 2, pp. 270 – 276, 1983. [Online]. Available: https://doi.org/10.1214/aop/1176993596