Asymptotic Behaviour of the Connection Probability of Cell Partitioned Networks

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Abstract

This report considers the problem of finding the asymptotic connectivity probability of cell partitioned networks. We specialize on the cases of 2D and 3D square and cubic networks. For the square 2D networks, we fully characterize the relationship between the asymptotic connectivity probability and the cell density. For 3D networks we partially characterize the above relationship.

I. INTRODUCTION

Consider a family of general undirected graphs $\{(\mathcal{G}_c, \mathcal{E}_c) : c \in \{1, 2, \dots\}\}$, where $(\mathcal{G}_c, \mathcal{E}_c)$, comprises of the set of c nodes $\mathcal{G}_c$ and edges $\mathcal{E}_c$. Assume, each node of the graph $\mathcal{G}_c$ can be independently switched on with probability p_c. The graph $\mathcal{G}_c$ is said to be connected if at least one node in the graph is switched on, and for any two nodes i, j that are switched on, there exists a collection of switched on nodes $i = a_0, a_1, \dots, a_t = j$, such that $(a_k, a_{k+1}) \in \mathcal{E}_c$ for all $k \in \{0, \dots, t-1\}$. The problem is to find $\lim_{c\to\infty} P[\mathcal{G}_c$ is connected$]$. This can be also seen as the problem of full-connectivity of the site percolation model [1].

A closely related model to the above is the random graph with edge percolation for the edge percolation model. Here the random graph $\mathcal{G}_{n,p}$ had n nodes and has no edges initially. We add an undirected edge between each pair of nodes independently with probability p. The work of [2] defines a class of statements called first order statements, where it is established that for any $p \in [0, 1]$ and any first order statement A, $\lim_{n\to\infty} \Pr[G_{n,p}$ has $A] = 0$ or 1 [2]. The full connectivity of the graph is one such first order statement. In addition to the above, it is established that there exists a critical function $p : \mathbb{N} \to [0, 1]$ such that when a function $r : \mathbb{N} \to [0, 1]$ grows asymptotically faster than p, then $\lim_{n\to\infty} \Pr[G_{n,r(n)}$ is connected $] = 1$ and

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when r grows asymptotically slower than p, $\lim_{n\to\infty} \Pr[G_{n,r(n)}$ is connected $] = 0$. This critical value is found to $p(n) = \ln(n)/n$ [2]. This behavior of existence of a critical function is also known as the phase transition behavior.

Graphs with a fixed radius model have also been considered [3]. Here, it is assumed that a node is connected to all the nodes that are within given radius of the node. Similar to above are graphs defined on grids [1]. These can be seen as a combination of the fixed radius model and the probabilistic model. Here, the graph $\mathcal{G}_{n,p}$ defined before takes the form of a grid. Instead of adding edge between each pair of nodes with probability p, we add an edge independently with probability p only between the pairs of nodes that are neighboring in the original graph (Two nodes are neighboring if they share an edge). In this case, it can be shown that when a function r grows asymptotically faster than $p(n) = \left(1 - \frac{1}{n^{1/4}}\right)$, we have that $\lim_{n\to\infty} \Pr[G_{n,r(n)}$ is connected $] = 1$, and when r grows asymptotically slower than p, we have that $\lim_{n\to\infty} \Pr[G_{n,r(n)}$ is connected $] = 0$. Hence, this case also exhibits a phase transition behavior.

All the above work consider full connectivity in edge percolation model and hence cannot be used to solve our original problem of interest which considers site percolation. Work on full connectivity with site percolation is sparse. It is hard to analyze the original problem for a general graph. Hence, we consider two special cases. First, we assume that the graph $\mathcal{G}_c$ is a 2D square grid with $c = k \times k$ nodes, as shown in Figure 1. Two nodes are connected if and only if they share at least one common vertex. For this model, we show that when p_c grows faster than $\left(1 - \frac{1}{c^{1/8}}\right)$, $\lim_{c\to\infty} P[\mathcal{G}_c$ is connected$] = 1$. The work of [4] shows that when p_c grows slower than $\left(1 - \frac{1}{c^{1/8}}\right)$, $\lim_{c\to\infty} P[\mathcal{G}_c$ is connected$] = 0$. Hence, we can observe a phase transition for the above graph. We initially prove using a constructive argument that when, p_c grows faster than $\left(1 - \frac{1}{c^{1/3}}\right)$, $\lim_{c\to\infty} P[\mathcal{G}_c$ is connected$] = 1$, after which we prove the full bound when p_c grows faster than $\left(1 - \frac{1}{c^{1/8}}\right)$. Next, we extend the above work to a 3D cell partitioned network, where the graph and the connections are defined similar to the 2D model. We establish that, when p_c grows faster than $\left(1 - \frac{1}{c^{2/3}}\right)$, $\lim_{c\to\infty} P[\mathcal{G}_c$ is connected$] = 1$ and when p_c grows slower than $\left(1 - \frac{1}{c^{1/26}}\right)$, $\lim_{c\to\infty} P[\mathcal{G}_c$ is connected$] = 0$.

Use cases of the above model include Wireless ad-hoc networks [5], social media networks [6], and disease spreading [7]

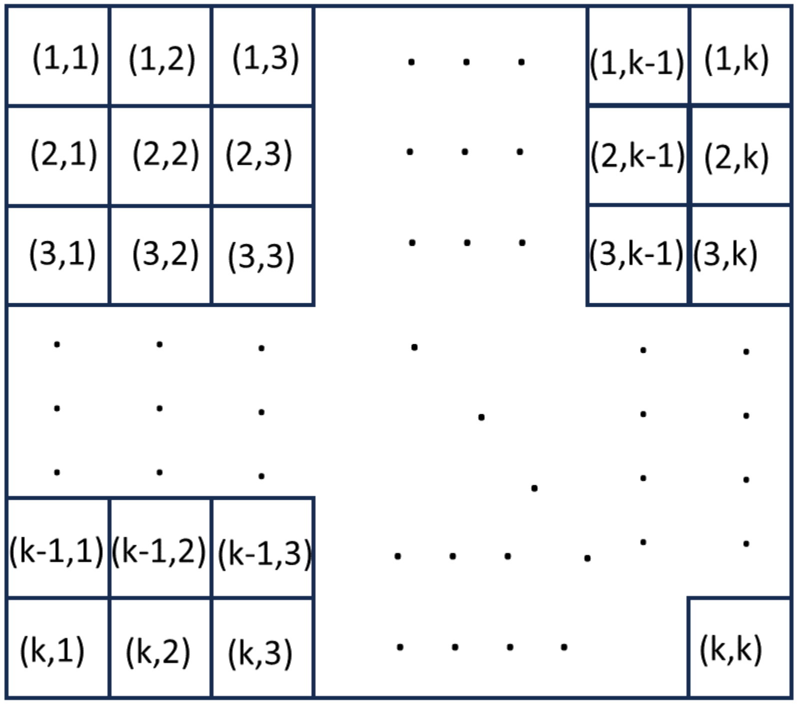

Fig. 1. 2D-square grid

A. Cell partitioned network model

The above 2D and 3D networks can be equivalently represented using the idea of cell partitioned networks. For the 2D case, the whole network has unit area, and is partitioned into $c = k \times k$ cells. Users are distributed in the network according to a Poisson Point Process with rate $\lambda = \theta c \log(c)$, where θ is a fixed constant. A cell is occupied/on if there is at least one user in the cell. Hence, each node is independently occupied with probability $\left(1 - \frac{1}{c^{\theta}}\right)$. We assume that a user can connect to another user in the network, only if there is a path of occupied cells from one user to the other. We say the network is connected if all the users are connected to each other. Hence, our analysis will establish that the network is asymptotically connected with probability 1, if $\theta > 1/8$, and is asymptotically disconnected with probability 1 if $\theta < 1/8$. We formulate the problem using this idea of cell partitioned networks, due to it's direct real world applicability. For certain cases such as for the proof of $\theta > 1/8$, we will use the formulation introduced before.

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II. 2D-MODEL

We first prove that the network is asymptotically connected with probability 1, if $\theta > 1/3$ using an intuitive constructive argument. Proving the full result for $\theta > 1/8$ is difficult to accomplish using a constructive argument. Rather, this proof requires certain results regarding the convergence of a sequence of random variables to a Poisson distribution.

A. Easier approach to for $\theta > 1/3$

We begin with the following lemma.

Lemma 1: Consider a 4×4 grid. Assume at least 14 cells in the grid are occupied. Then, the grid is connected.

Proof: We will consider the case when exactly 14 cells are occupied. The rest will trivially follow from this case.

Let us name the rows of the 4×4 grid from 1 to 4 and the columns of the 4×4 grid from 1 to 4. Notice that from the pigeon hole principle, there exist two columns with all cells occupied. Let these columns be i and j

Case 1 $(i, j) = (1, 3)$ or $(i, j) = (2, 4)$: Notice that both cases are similar. Hence, we will only consider $(i, j) = (1, 3)$. Notice that in this case, at least one occupied cell exists in column 2. This cell will connect columns 1 and 3. On the other hand, any occupied cell in columns 2 or 4 is directly connected to either column 1 or 3. Hence, the grid is connected.

Case 2 $(i, j) = (1, 4)$: Notice that there are six occupied cells in columns 2 and 3. Hence, at least two occupied cells exist in both columns 2 and 3. Hence, an occupied cell exists in column two that is directly connected to an occupied cell in column 3. These two cells will connect columns 1 and 4. Also, any occupied cell in columns 2 or 3 is connected to either column 1 or 4. Hence, the grid is connected.

Case 3 $(i, j) = (2, 3)$: Notice that columns 2 and 3 are connected. Also, any occupied cell in columns 1 or 4 is directly connected to either column 2 or 3. Hence, the grid is connected.

Case 4 $(i, j) = (1, 2)$ or $(i, j) = (3, 4)$: Notice that both cases are similar. Hence, we will only consider $(i, j) = (1, 2)$. Notice that columns 1 and 2 are connected. Also, any occupied cell in column 3 will be directly connected to column 2. Now consider an occupied cell (x, y) in column 4. Hence, we have $y = 4$. If $x \in \{2, 3\}$, notice that at least one of the cells $(x - 1, 3)$, $(x, 3)$ or $(x + 1, 3)$ will be occupied. Hence, (x, y) will be directly connected to an occupied cell in column 3. If $x = 1$, notice that (x, y) will not be directly connected to an occupied cell in column 3 if and only if both $(1, 3)$, and $(2, 3)$ are not occupied. In that case, $(2, 4)$ has to be occupied. From the previous argument $(2, 4)$ is directly connected to an occupied cell in column 3. Hence, (x, y) will be connected to an occupied cell in column 3. Similarly, if $x = 4$, we have that (x, y) will be connected to an occupied cell in column 3. Hence, any occupied cell in column 4 will be connected to an occupied cell in column 3. Hence, the grid is connected. ■

Consider the original cell partitioned network with $k = 4m$ for some integer m. Hence, $C = 16m^2$. Now subdivide the $4m \times 4m$ grid into m^2 sub-grids each of size 4×4. Assume that each of the 4×4 sub-grids contains at least 14 occupied cells. Then from Lemma 1, the occupied cells within each sub-grid will be connected. Now we establish that any such sub-grid will be connected to the sub-grids that are adjacent to it. This will establish that the network will be connected. For this, we establish that any sub-grid that is not in the last row of sub-grids is connected to the sub-grid immediately below it. This will establish that a sub-grid is connected to a sub-grid adjacent to it from symmetry.

Lemma 2: Assume the configuration defined in the previous paragraph. Then any sub-grid that is not in the last row of sub-grids is connected to the sub-grid immediately below it.

Proof: Consider a sub-grid A, that is not in the last row of sub-grids. We prove that A is connected to the sub-grid B, which is immediately below A. Note that since A contains at least 14 occupied cells, the last row of A will contain at least 2 occupied cells. Similarly, the top row of B will contain at least 2 occupied cells. Hence, an occupied cell exists in the last row of A that is directly connected to an occupied cell in the top row of B. This will connect the two sub-grids. ■

Notice that due to the above for the $4m \times 4m$ grid we have that,

$$\begin{aligned} &P[\text{Network is connected}] \\ &\qquad \geq P[\text{there are at least 14 occupied cells in each } 4 \times 4 \text{ sub-grid}] \\ &\qquad =_{(a)} P[\text{there are at least 14 occupied cells in the first } 4 \times 4 \text{ sub-grid}]^{m^2}, \end{aligned} \tag{1}$$

where (a) follows since the distribution of users in each sub-grid is i.i.d. due to the Poisson point process. Note that,

$$P[\text{there are at least 14 occupied cells in the first } 4 \times 4 \text{ sub-grid}]$$

$$
\begin{aligned}
&= 1 - \left(\sum_{i=0}^{13} P[\text{first } 4 \times 4 \text{ sub-grid has exactly } i \text{ occupied cells}] \right) \\
&= 1 - \left(\sum_{i=1}^{13} \binom{16}{i} \left(1 - e^{-\theta \log(c)}\right)^i \left(e^{-\theta \log(c)}\right)^{16-i} \right) \\
&= 1 - e^{-3\theta \log(c)} \left(\sum_{i=1}^{13} \binom{16}{i} \left(1 - e^{-\theta \log(c)}\right)^i \left(e^{-\theta \log(c)}\right)^{13-i} \right) \\
&= 1 - e^{-3\theta \log(c)} \left(\sum_{i=1}^{13} \frac{16 \times 15 \times 14}{(16-i) \times (15-i) \times (14-i)} \binom{13}{i} \left(1 - e^{-\theta \log(c)}\right)^i \left(e^{-\theta \log(c)}\right)^{13-i} \right) \\
&\geq 1 - e^{-3\theta \log(c)} \left(\sum_{i=1}^{13} \frac{(16 \times 15 \times 14)}{6} \binom{13}{i} \left(1 - e^{-\theta \log(c)}\right)^i \left(e^{-\theta \log(c)}\right)^{13-i} \right) \\
&= 1 - 560 e^{-3\theta \log(c)} \left(1 - e^{-\theta \log(c)} + e^{-\theta \log(c)}\right)^{13} \\
&= 1 - 560 e^{-3\theta \log(c)} \qquad (2)
\end{aligned}
$$

Hence, substituting in (1), for large enough m we have that,

$$
P[\text{Network is connected}] \geq \left(1 - 560 e^{-3\theta \log\left(16m^2\right)}\right)^{m^2} = \left(1 - \frac{560}{(16m^2)^{3\theta}}\right)^{m^2} \qquad (3)
$$

Hence, for $\theta > 1/3$, taking limit as m tends to infinity, we have that, $P[\text{Network is connected}]$ approaches 1 as desired.

Remark: Notice that we only proved that the network connectivity probability approaches 1 when we consider a sequence of networks of the form $4m \times 4m$ where m approaches ∞. To establish the result for a general sequence $k \times k$ networks where k approaches infinity we need little more work. In particular, we need to treat the corners of the grid in a different manner. We omit these details for brevity.

B. Approach for $\theta > 1/8$

In this section, we focus on establishing that when $\theta > 1/8$ the network is asymptotically connected with probability 1. Notice that, each cell in the network is occupied with probability $p_k = \left(1 - \frac{1}{k^{2\theta}}\right)$. For notational convenience, we will use $\mathcal{G}_k$ to denote the graph with $k \times k$ cells hereafter, whereas from our original definition, the graph should be names as $\mathcal{G}_{k^2}$.

Claim 1: We prove a stronger statement. Consider a $k \times k$ cell partitioned network $\mathcal{G}_k$ where each cell is independently occupied with probability p_k where $p_k = \left(1 - \frac{d_k}{k^{(1/4)}}\right)$, where it is known that $\lim_{k \to \infty} d_k = 0$. Then $\lim_{k \to \infty} P(\mathcal{G}_k \text{ is connected}) = 1$.

Before moving on to the proof of the claim we begin with a preliminary lemma.

Lemma 3: Consider indicator random variables I_α for $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is a finite abstract set. Let $\mathbb{E}\{I_\alpha\} = p_\alpha$, and let,

$$\lambda = \sum_{\alpha \in \mathcal{A}} p_\alpha \tag{4}$$

Also assume that there exists a finite collection of independent random variables $X = \{X_1, X_2, \ldots, X_n\}$ such that each I_α is a non-decreasing function of random variables in X. Let $W = \sum_{\alpha \in \mathcal{A}} I_\alpha$. Then we have,

$$d_{TV}(W, \text{Po}(\lambda)) \leq \frac{1 - e^{-\lambda}}{\lambda} \left(\text{Var}(W) - \lambda + 2 \sum_{\alpha \in \mathcal{A}} p_\alpha^2 \right), \tag{5}$$

where $d_{TV}(A, B)$ is the total variation distance between two random variables A and B (For more details on Poisson approximation and bounds like above, see [8]).

Let us call a cell (doesn't necessarily need to be occupied) isolated if all it's neighbors are not occupied. Now we prove the following lemma regarding isolated cells.

Lemma 4: Let W_k denote the number of isolated cells of $\mathcal{G}_k$. Then W_k converges in distribution to a Poisson random variable with parameter λ if $k^2(1 - p_k)^8$ converges to λ.

Proof: Notice that

$$W_k = \sum_{i=1}^{k} \sum_{j=1}^{k} I_{i,j}, \tag{6}$$

where $I_{i,j}$ is an indicator random variable indicating whether the cell (i, j) is isolated or not. Notice that each $\mathcal{I}_{i,j}$ is an non-decreasing function of the random variables $\{X_{i,j} : i, j \in \{1, \ldots, k\}\}$, where,

$$X_{i,j} = \begin{cases} 1 & \text{if cell } (i,j) \text{ is empty} \\ 0 & \text{if cell } (i,j) \text{ is occupied} \end{cases} \tag{7}$$

Define, $\mathbb{E}\{I_{i,j}\} = p_{i,j}$, and

$$\lambda_k = \mathbb{E}\{W_k\} = \sum_{i=1}^{k} \sum_{j=1}^{k} p_{i,j}, \tag{8}$$

Hence, we can apply Lemma 3, to obtain,

$$d_{TV}(W_k, \text{Po}(\lambda_k)) \leq \frac{1 - e^{-\lambda_k}}{\lambda_k} \left(\text{Var}(W_k) - \lambda_k + 2 \sum_{i=1}^{k} \sum_{j=1}^{k} p_{i,j}^2 \right), \tag{9}$$

To prove the lemma, we prove that the right-hand side of the above expression goes to zero as $k \to \infty$ and that $\lambda_k \to \lambda$ as $k \to \infty$. This would imply that as $k \to \infty$, both $d_{TV}(W_k, \text{Po}(\lambda_k))$, and $d_{TV}(\text{Po}(\lambda_k), \text{Po}(\lambda))$ go to zero. Hence, we have that as $k \to \infty$, $d_{TV}(W_k, \text{Po}(\lambda))$ goes to zero which establishes the result.

We start by calculating $p_{i,j}$ for each i, j. Notice that,

$$p_{i,j} = \begin{cases} (1-p_k)^3 \text{ if } (i,j) \text{ is a corner vertex} \\ (1-p_k)^5 \text{ if } (i,j) \text{ is an edge vertex but not a corner vertex} \\ (1-p_k)^8 \text{ otherwise} \end{cases} \tag{10}$$

Notice that we require the values of the quantities $\text{Var}(W_k)$, and $\sum_{i=1}^{k}\sum_{j=1}^{k} p_{i,j}^2$ as k goes to infinity. We start by calculating $\mathbb{E}\{W_k\}$. Notice that,

$$\lambda_k = \mathbb{E}\{W_k\} = 4(1-p_k)^3 + 4(k-2)(1-p_k)^5 + (k-2)^2(1-p_k)^8 \to \lambda$$

where the last limit follows from the fact that $k^2(1-p_k)^8$ converges to λ. Now notice that,

$$\sum_{i=1}^{k}\sum_{j=1}^{k} p_{i,j}^2 = 4(1-p_k)^6 + 4(k-2)(1-p_k)^{10} + (k-2)^2(1-p_k)^{16} \to 0 \tag{11}$$

Now we calculate $\mathbb{E}\left\{W_k^2\right\}$. Notice that,

$$\mathbb{E}\left\{W_k^2\right\} = \mathbb{E}\left\{\left(\sum_{i=1}^{k}\sum_{j=1}^{k} I_{i,j}\right)\left(\sum_{a=1}^{k}\sum_{b=1}^{k} I_{a,b}\right)\right\} = \sum_{(i,j)\in\mathcal{G}_k}\sum_{(a,b)\in\mathcal{G}_k} \mathbb{E}\{I_{i,j}I_{a,b}\} \tag{12}$$

Hence, we require, $\mathbb{E}\{I_{i,j}I_{a,b}\}$. Notice that $I_{i,j}I_{a,b} = 1$ if and only if both nodes (i,j) and (a,b) are isolated. This will be true if and only if the neighbors of (i,j) are not occupied and the neighbors of (a,b) are not occupied.

For a given k, and node $(i,j) \in \mathcal{G}_k$, define the square $S_{i,j}^k$ to be the set,

$$S_{i,j}^k = \{(a,b) \in \mathcal{G}_k : \ |a-i| \le 2, \ |b-j| \le 2\}. \tag{13}$$

Notice that $(a,b) \in S_{i,j}^k$ if and only if $(i,j) \in S_{a,b}^k$. We define the equivalence relation $\sim$ as $(i,j) \sim (a,b)$ iff $(a,b) \in S_{i,j}^k$. Notice that,

$$\begin{aligned}\sum_{(i,j)\in\mathcal{G}_k}\sum_{(a,b)\in\mathcal{G}_k} \mathbb{E}\{I_{i,j}I_{a,b}\} &= \sum_{(i,j)\in\mathcal{G}_k} \mathbb{E}\{I_{i,j}\} + \sum_{\substack{(i,j)\sim(a,b)\\(i,j)\neq(a,b)}} \mathbb{E}\{I_{i,j}I_{a,b}\} + + \sum_{(i,j)\not\sim(a,b)} \mathbb{E}\{I_{i,j}I_{a,b}\} \\ &= \lambda + \sum_{\substack{(i,j)\sim(a,b)\\(i,j)\neq(a,b)}} \mathbb{E}\{I_{i,j}I_{a,b}\} + + \sum_{(i,j)\not\sim(a,b)} \mathbb{E}\{I_{i,j}I_{a,b}\} \end{aligned} \tag{14}$$

We define the three sets $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, where $\mathcal{A}$ is the set of corner points, $\mathcal{B}$ is the set of boundary points that are not corner points and $\mathcal{C}$ is the set of interior points. Also define $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$. Notice that,

$$\begin{aligned}\sum_{(i,j)\not\sim(a,b)} \mathbb{E}\{I_{i,j}I_{a,b}\} = & \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{A} \text{ and } (a,b)\in\mathcal{A}}} \mathbb{E}\{I_{i,j}I_{a,b}\} + 2 \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{A} \text{ and } (a,b)\in\mathcal{B}}} \mathbb{E}\{I_{i,j}I_{a,b}\} \\ & + 2 \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{A} \text{ and } (a,b)\in\mathcal{C}}} \mathbb{E}\{I_{i,j}I_{a,b}\} + \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{B} \text{ and } (a,b)\in\mathcal{B}}} \mathbb{E}\{I_{i,j}I_{a,b}\} \\ & + 2 \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{B} \text{ and } (a,b)\in\mathcal{C}}} \mathbb{E}\{I_{i,j}I_{a,b}\} \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{C} \text{ and } (a,b)\in\mathcal{C}}} \mathbb{E}\{I_{i,j}I_{a,b}\} \qquad (15)\end{aligned}$$

We establish that all the sums above except the last one converge to zero as k goes to infinity.

Notice that when $(i,j) \not\sim (a,b)$ we have that $I_{i,j}$ and $I_{a,b}$ are independent.

$$\sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{A} \text{ and } (a,b)\in\mathcal{A}}} \mathbb{E}\{I_{i,j}I_{a,b}\} = \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{A} \text{ and } (a,b)\in\mathcal{A}}} \mathbb{E}\{I_{i,j}\}\mathbb{E}\{I_{a,b}\} = \binom{4}{2}(1-p_k)^6 \to 0, \qquad (16)$$

$$\sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{A} \text{ and } (a,b)\in\mathcal{B}}} \mathbb{E}\{I_{i,j}I_{a,b}\} = \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{A} \text{ and } (a,b)\in\mathcal{B}}} \mathbb{E}\{I_{i,j}\}\mathbb{E}\{I_{a,b}\} \leq 4 \times (4k-8)(1-p_k)^8 \to 0, \qquad (17)$$

where the inequality follows since, $|\mathcal{A}| = 4$, and $|\mathcal{B}| = 4k-8$, and the last inequality follows from the fact that $k^2(1-p_k)^8$ converges to λ. Also,

$$\sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{A} \text{ and } (a,b)\in\mathcal{C}}} \mathbb{E}\{I_{i,j}I_{a,b}\} = \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{A} \text{ and } (a,b)\in\mathcal{C}}} \mathbb{E}\{I_{i,j}\}\mathbb{E}\{I_{a,b}\} \leq 4k^2(1-p_k)^{11} \to 0 \qquad (18)$$

where the inequality follows since, $|\mathcal{C}| \leq k^2$, and $|\mathcal{A}| = 4$, and the last inequality follows from the fact that $k^2(1-p_k)^8$ converges to λ. Now,

$$\sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{B} \text{ and } (a,b)\in\mathcal{B}}} \mathbb{E}\{I_{i,j}I_{a,b}\} = \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{B} \text{ and } (a,b)\in\mathcal{B}}} \mathbb{E}\{I_{i,j}\}\mathbb{E}\{I_{a,b}\} \leq (4k-8)^2(1-p_k)^{10} \qquad (19)$$

where the inequality follows since, $|\mathcal{B}| = 4k-8$, and the last inequality follows from the fact that $k^2(1-p_k)^8$ converges to λ. Finally,

$$\sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{B} \text{ and } (a,b)\in\mathcal{C}}} \mathbb{E}\{I_{i,j}I_{a,b}\} = \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{B} \text{ and } (a,b)\in\mathcal{C}}} \mathbb{E}\{I_{i,j}\}\mathbb{E}\{I_{a,b}\} \leq (4k-8)k^2(1-p_k)^{13} \qquad (20)$$

where the inequality follows since, $|\mathcal{C}| \leq k^2$, and $|\mathcal{B}| = 4k-8$, and the last inequality follows from the fact that $k^2(1-p_k)^8$ converges to λ.

Now we focus on calculating the limit of the last term in (15). Notice that

$$\sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{C}\text{ and }(a,b)\in\mathcal{C}}} \mathbb{E}\{I_{i,j}I_{a,b}\} = A(1-p_k)^{16}, \tag{21}$$

where A is the number of pairs, $[(i,j),(a,b)]$ such that $(i,j),(a,b)\in\mathcal{C}$ such that $(i,j)\not\sim(a,b)$. Fix $(i,j)\in\mathcal{C}$. Notice that there are at least $(k-2)^2-25$ nodes in $\mathcal{C}$ that are not related to (i,j). Hence, $A \geq (k-2)^2((k-2)^2-25)$. Also, we trivially have $A \leq (k-2)^4$. Hence,

$$[(k-2)^2((k-2)^2-25)](1-p_k)^{16} \leq \sum_{\substack{(i,j)\not\sim(a,b)\\(i,j)\in\mathcal{C}\text{ and }(a,b)\in\mathcal{C}}} \mathbb{E}\{I_{i,j}I_{a,b}\} = A(1-p_k)^{16} \leq (k-2)^4(1-p_k)^{16}. \tag{22}$$

Notice that as $k\to\infty$ both $[(k-2)^2((k-2)^2-25)](1-p_k)^{16}$ and $(k-2)^4(1-p_k)^{16}$ go to λ^2. Hence, we have from (15) that,

$$\sum_{(i,j)\not\sim(a,b)} \mathbb{E}\{I_{i,j}I_{a,b}\} \to \lambda^2 \tag{23}$$

Now, we establish that,

$$\sum_{\substack{(i,j)\sim(a,b)\\(i,j)\neq(a,b)}} \mathbb{E}\{I_{i,j}I_{a,b}\} \to 0 \tag{24}$$

which will establish that

$$\mathbb{E}\left\{W_k^2\right\} \to \lambda^2+\lambda \tag{25}$$

from (14) and (12). Notice that,

$$\sum_{\substack{(i,j)\sim(a,b)\\(i,j)\neq(a,b)}} \mathbb{E}\{I_{i,j}I_{a,b}\} = \sum_{\substack{(i,j)\sim(a,b)\\(i,j)\in\mathcal{D}\text{ and }(a,b)\in\mathcal{D}}} \mathbb{E}\{I_{i,j}I_{a,b}\} + \sum_{\substack{(i,j)\sim(a,b)\\(i,j)\notin\mathcal{D}\text{ or }(a,b)\notin\mathcal{D}}} \mathbb{E}\{I_{i,j}I_{a,b}\} \tag{26}$$

Now it is easy to see that if $(i,j)\sim(a,b)$ and $(i,j),(a,b)\in\mathcal{D}$, then there are at least 6 cells that neighbor either (i,j) or (a,b). Also, notice that there are at most $20k$ pairs $[(i,j),(a,b)]$ and $(i,j),(a,b)\in\mathcal{D}$ and $(a,b)\sim(i,j)$. Hence,

$$\sum_{\substack{(i,j)\sim(a,b)\\(i,j)\in\mathcal{D}\text{ and }(a,b)\in\mathcal{D}}} \mathbb{E}\{I_{i,j}I_{a,b}\} \leq 20k(1-p_k)^6 \to 0. \tag{27}$$

Also, it is easy to see that if $(i,j) \sim (a,b)$ and at least one of (i,j) and (a,b) belong to $\mathcal{C}$, then there are at least 9 cells that neighbor either (i,j) or (a,b). Also, notice that there are at most $25(k-2)^2$ pairs $[(i,j),(a,b)]$ and $(i,j),(a,b) \in \mathcal{C}$ and $(a,b) \sim (i,j)$. Hence,

$$\sum_{\substack{(i,j)\sim(a,b) \\ (i,j)\in\mathcal{D} \text{ and } (a,b)\in\mathcal{D}}} \mathbb{E}\{I_{i,j}I_{a,b}\} \leq 25(k-2)^2(1-p_k)^9 \to 0. \tag{28}$$

Hence, we are done. Hence, we have that, $\mathbb{E}\{W_k^2\} \to \lambda^2 + \lambda$.

Substituting the above and (11) in (9), we have that $d_{TV}(W_k, \text{Po}(\lambda_k)) \to 0$ as $k \to \infty$. Hence, the lemma is proved. ■

Now, we also have the following lemma.

Lemma 5: Assume that $p_k = \left(1 - \frac{d_k}{k^{1/4}}\right)$, and $d_k \to d \in [0,\infty)$. Consider the event $A_k = \{\mathcal{G}_k \text{ is made of isolated nodes, unoccupied nodes, and one connected component}\}$. We have that $\lim_{k\to\infty} P[A_k] = 1$.

Proof: Notice that

$$\begin{aligned} A_k^c = &\{\mathcal{G}_k \text{ contains at least two distinct connected components with each} \\ &\quad \text{component having at least two nodes.}\} \end{aligned} \tag{29}$$

We consider two cases. Consider the above complement event. Let C and B be the two disjoint connected components.

Case 1: At least one of C and B do not have nodes in the edge of $\mathcal{G}_k$

Assume without loss of generality that C is completely contained in the interior of $\mathcal{G}_k$. Then there should be at least 10 unoccupied nodes surrounding C. Let $\varepsilon_{i,j,t}$ denote the event of having an unoccupied path with t nodes starting from node (i,j). Notice that, $P[\varepsilon_{i,j,t}] \leq 8^{t-1}(1-p_k)^t$ which follows from a simple union bound since each node neighbours at most 8 nodes. Now notice that,

$$\begin{aligned} &P[A^c, \text{ at least one of } C \text{ and } B \text{ belong to the interior of } \mathcal{G}_k] \\ &\quad \leq P[\text{there is an unoccupied path of length } 10] \leq \sum_{i=1}^{k}\sum_{j=1}^{k} P[\varepsilon_{i,j,10}] \leq k^2 8^9 (1-p_k)^9 \end{aligned} \tag{30}$$

Hence, $P[A^c, \text{ at least one of } C \text{ and } B \text{ belong to the interior of } \mathcal{G}_k] \to 0$ as $k \to \infty$.

Case 2: Both C and B have nodes in the edge of $\mathcal{G}_k$.

This means that both C and B have a node that belongs to the boundary. Notice that unless one of C or B is a component with two occupied nodes where the two nodes are a corner node and the node immediately below/above/left/right of it, there should be an unoccupied path of at least 5 nodes, starting from the boundary of $\mathcal{G}_k$ that separates the two components. In the special event mentioned above, which we call event L, there has to be an unoccupied path of length 4. Notice that there are 8 possibilities for this path and each has a probability $(1-p_k)^4$ (One path for each connected component with two occupied cells, where the two nodes are a corner node and the node immediately below/above/left/right of it). Hence $P(L) \leq 8(1-p_k)^4$

$$\begin{aligned} &P[A^c, \text{ both } A \text{ and } B \text{ have a node in the edge of } \mathcal{G}_k] \\ &\quad = P[A^c, \text{ both } A \text{ and } B \text{ have a node in the edge of } \mathcal{G}_k, L] \\ &\qquad + P[A^c, \text{ both } A \text{ and } B \text{ have a node in the edge of } \mathcal{G}_k, L^c] \\ &\quad = P(L) + P[\text{there is an unoccupied path of length } 5 \text{ starting from an edge node}] \\ &\quad \leq 8(1-p_k)^4 + \sum_{(i,j)\in\mathcal{D}} P[\varepsilon_{i,j,4}] \leq 8(1-p_k)^4 + 4k \times 8^4(1-p_k)^5, \end{aligned} \tag{31}$$

where the last inequality follows since there are at most $4k$ nodes in the boundary $\mathcal{D}$, and from a simple union bound similar to the previous case. Notice that since $k^2(1-p_k)^8 \to \lambda$, it can be easily seen that the above goes to zero as $k \to \infty$.

Combining the two claims, we have $P[A^c]$ goes to zero as $k \to \infty$. Hence, we are done. ■

Now we are ready to prove the main claim of this section. We will rewrite the claim here for brevity.

Claim 1: Consider a $k \times k$ cell partitioned network $\mathcal{G}_k$ where each cell is independently occupied with probability p_k where $p_k = \left(1 - \frac{d_k}{k^{(1/4)}}\right)$, where it is known that $\lim_{k\to\infty} d_k = 0$. Then $\lim_{k\to\infty} P(\mathcal{G}_k \text{ is connected}) = 1$.

Proof: Fix $\lambda > 0$. Consider a sequence $\{q_k : k \in \mathbb{N}\}$ such that $k^2(1-q_k)^8 \to \lambda$. Notice that from the given condition in the claim, p_k grows faster than q_k. Hence there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $q_k \leq p_k$. Let $\mathcal{H}_k$ denote the graph generated when q_k is used as the probability of cell occupation. Also, let V_k denote the number of isolated cells of $\mathcal{H}_k$. Notice that since $p_k \geq q_k$, we should have, $P[W_k = 0] \geq P[V_k = 0]$

Let us define the events

$$S_k = \{\mathcal{G}_k \text{ does not have two connected components with each having at least 2 cells}\} \tag{32}$$

Now notice that,

$$
\begin{aligned}
P[\mathcal{G}_k \text{ is connected}] &\geq P[\mathcal{G}_k \text{ has no isolated cells}, S_k] \\
&\geq P[W_k = 0, S_k] \geq P[W_k = 0] + P[S_k] - 1 \\
&\geq P[V_k = 0] + P[S_k] - 1 \qquad (33)
\end{aligned}
$$

Taking the $\liminf$ as k goes to infinity,

$$
\liminf_{k \to \infty} P[\mathcal{G}_k \text{ is connected}] \geq \liminf_{k \to \infty} P[V_k = 0] + P[S_k] - 1 =_{(a)} e^{-\lambda} + 1 - 1 = e^{-\lambda} \qquad (34)
$$

where (a) follows due to Lemma 4 and Lemma 5. Notice that our choice of $\lambda > 0$ was arbitrary. Hence, we have, Hence,

$$
\liminf_{k \to \infty} P[\mathcal{G}_k \text{ is connected}] \geq e^0 = 1. \qquad (35)
$$

Hence, we are done. ■

III. 3-D NETWORKS

Now we extend the work on the 2D cell partitioned network to a 3D cell partitioned. Similar to the 2D model, we assume that a unit cube is divided into $c = k^3$ cells, and wireless users are distributed according to a Poisson point process of intensity $\lambda = \theta c \ln(c)$. The definition of connectivity is similar to the 2D case, where it is assumed that each cell neighbours the cells that either share a vertex, edge or a face with the particular cell. Figures 2-4 examples of three such networks with different values of θ.

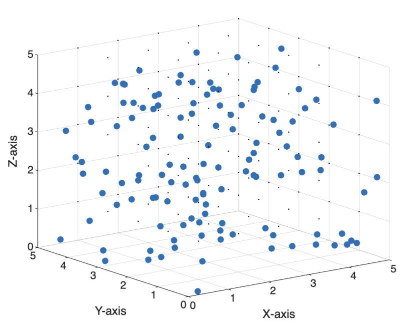

Fig. 2. Network 1

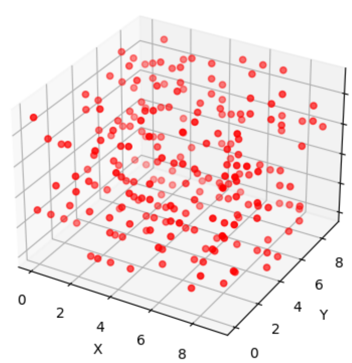

Fig. 3. Network 2 ($\theta = 1/26$)

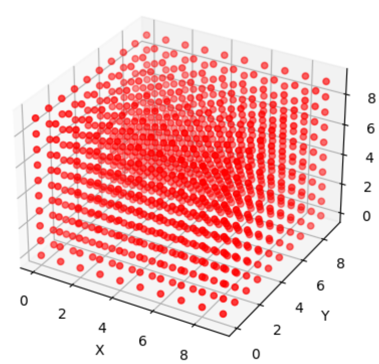

Fig. 4. Network 3 ($\theta = 1$)

A. Approach for $\theta < 1/26$

We first consider the case $\theta < 1/26$, and prove that $\lim_{k\to\infty} P[\text{Network is connected}] = 0$ for this case. For this assume $k = 3s$ ($c = (3s)^3$) for some positive integer s, so that the network region is a cubic grid with side length $3s$. Group the c cells into s^3 disjoint $3 \times 3 \times 3$ blocks. For each block $i \in \{1, \ldots, s^3\}$, define the event E_i as the event that block i has a disconnection event where the middle cell of block i has at least one user and the 26 surrounding cells are empty. Hence,

$$P[E_i] = \left(e^{-\lambda/c}\right)^{26}\left(1 - e^{-\lambda/c}\right) = e^{-26\lambda/c} - e^{-27\lambda/c}, \quad \forall i \in \{1, \ldots, s^3\} \tag{36}$$

If E_i^c denotes the complement of event E_i, then

$$P[E_i^c] = 1 - e^{-26\lambda/c} + e^{-27\lambda/c} = 1 - \frac{1}{c^{26\theta}} + \frac{1}{c^{27\theta}} \tag{37}$$

where the final equality uses $\lambda = \theta c \ln(c)$. Define the event F that there are at least two nonempty cells. The complement event F^c is the event that either all cells are empty, or exactly $c-1$ cells are empty. Hence,

$$\begin{aligned} P[F^c] &= e^{-\lambda} + c\left(e^{-\lambda/c}\right)^{c-1}\left(1 - e^{-\lambda/c}\right) \\ &= (1-c)e^{-\lambda} + ce^{-\lambda\left(\frac{c-1}{c}\right)} \\ &= \frac{1-c}{c^{\theta c}} + \frac{c}{c^{\theta(c-1)}} \end{aligned} \tag{38}$$

Hence, notice that for any $\theta > 0$ we have:

$$\lim_{c\to\infty} P[F^c] = 0 \tag{39}$$

Now, notice that, if at least one event E_i occurs for some $i \in \{1, \ldots, s^3\}$, and if event F occurs (so that there are at least two nonempty cells), then the network must be disconnected. Formally, we have that,

$$\begin{aligned} P[\text{Network is disconnected}] &\geq P\left[\left(\bigcup_{i=1}^{s^3} E_i\right) \cap F\right] \geq_{(a)} P\left[\bigcup_{i=1}^{s^3} E_i\right] - P[F^c] \\ &= 1 - P\left[\bigcap_{i=1}^{s^3} E_i^c\right] - P[F^c] \\ &=_{(b)} 1 - \left(1 - \frac{1}{c^{26\theta}} + \frac{1}{c^{27\theta}}\right)^{s^3} - P[F^c] \end{aligned}$$

$$=_{(c)} 1 - \left(1 - \frac{1}{c^{26\theta}} + \frac{1}{c^{27\theta}}\right)^{c/27} - P[F^c] \tag{40}$$

where (a) holds by the inequality $P[A \cap B] \geq P[A] - P[B^c]$ for any events A, B, equality (b) holds by (37) together with the fact that the events $\{E_i^c\}_{i=1}^{s^3}$ are mutually independent, and equality (c) holds because $c = 27s^3$. From a similar logic to the $\theta > 1/3$ case is Section II-A, it can be shown that if $\theta < \frac{1}{26}$ then

$$\lim_{c\to\infty} \left(1 - \frac{1}{c^{26\theta}} + \frac{1}{c^{27\theta}}\right)^{c/27} = 0 \tag{41}$$

Combining this with (40), we have that, if $\theta < 1/26$,

$$\lim_{c\to\infty} P[\text{Network is disconnected}] = 1 \tag{42}$$

B. Approach for $\theta > 2/3$

In this section, we prove that prove that $\lim_{k\to\infty} P[\text{Network is connected}] = 1$ whenever $\theta > 2/3$. For this fix $k = 2s$ ($c = (2s)^3$). Index the cells of the network by (a, b, d) for $a, b, d \in \{1, \dots, 2s\}$, where a indexes the row, b the column, and d the depth. Define L as the set of cells on the bottom (basement) of the network, i.e.

$$L = \{(a, d, 1) \mid a, d \in \{1, \dots, 2s\}\} \tag{43}$$

connectivity is ensured if all cells in the set L have at least one node and if all nodes in the network have a path to the set L. For each $i, j \in \{1, \dots, s\}$, define $R_{i,j}$, the (i, j)-th vertical block as the set,

$$R_{i,j} = \{(a, b, d) \mid a \in \{2i-1, 2i\}, b \in \{2j-1, 2j\}, d \in \{2, \dots, 2s\}\} \tag{44}$$

To better visualize the above sets, we can think of the grid as a small city built on the basement L. Here, the set $R_{i,j}$ is the (i, j) th building. Notice that for $i, j \in \{1, \dots, s\}$, and each $l \in \{2, \dots, 2s\}$ the set,

$$X_{(i,j,l)} = \{(a, b, l) \mid a \in \{2i-1, 2i\}, b \in \{2j-1, 2j\}\} \tag{45}$$

denotes the l-th floor of building (i, j). The idea is to observe that network connectivity is ensured if all floors of all buildings have at least one occupied node and all the cells in the basement L are occupied.

We define A as the event that all cells in the set L are occupied. Notice that,

$$P[A] = (1 - e^{-\lambda/c})^{4s^2} = \left(1 - \frac{1}{c^\theta}\right)^{c^{\frac{2}{3}}} \tag{46}$$

Define $B_{i,j}$ as the event of each floor of building $R_{i,j}$ having at least on occupied cell. Notice that,

$$P[B_{i,j}] = \left(1 - e^{-\lambda(4/c)}\right)^{(2s-1)} = \left(1 - e^{-\theta c \log(c)\left(\frac{4}{c}\right)}\right)^{(2s-1)} =_{(b)} \left(1 - \frac{1}{c^{4\theta}}\right)^{c^{\frac{1}{3}}-1} \tag{47}$$

Hence we have that,

$$\begin{aligned} P[\text{Network is Connected}] &\geq P[A, B_{i,j} \text{ for all } i,j \in \{1,2,\ldots,s\}] \\ &=_{(a)} P[A] \left(P[B_{ij}]\right)^{s^2} \\ &=_{(b)} \left(1 - \frac{1}{c^\theta}\right)^{c^{\frac{2}{3}}} \left(1 - \frac{1}{c^{4\theta}}\right)^{\frac{c}{4} - \frac{c^{2/3}}{4}} \\ &= \left(1 - \frac{1}{c^\theta}\right)^{c^\theta / c^{(\theta - 2/3)}} \left(\left(1 - \frac{1}{c^{4\theta}}\right)^{c^{4\theta}}\right)^{\frac{1}{4c^{4\theta-1}} - \frac{1}{4c^{4\theta-2/3}}} \end{aligned} \tag{48}$$

where (a) follows since the occupation cells are independent, and (b) follows from (46), and (47). Now, we have the following well known lemma.

Lemma 6: For any functions $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ that satisfy:

$$\lim_{c \to \infty} h(c) = \infty, \tag{49}$$

$$\lim_{c \to \infty} g(c) = 0, \tag{50}$$

we have

$$\lim_{c \to \infty} \left(1 - \frac{1}{h(c)}\right)^{h(c)g(c)} = 1 \tag{51}$$

Using the previous lemma on (48), we have that, $\lim_{c \to \infty} P[\text{Network is Connected}] = 1$, whenever $\theta > 2/3$, since when $\theta > 2/3$, we have $\frac{1}{c^{(\theta - 2/3)}} \to 0$, and $\frac{1}{4c^{4\theta-1}} - \frac{1}{4c^{4\theta-2/3}} \to 0$. Hence, we have that,

$$\lim_{c \to \infty} P[\text{Network is Connected}] = 1, \tag{52}$$

whenever $\theta > 2/3$, as desired.

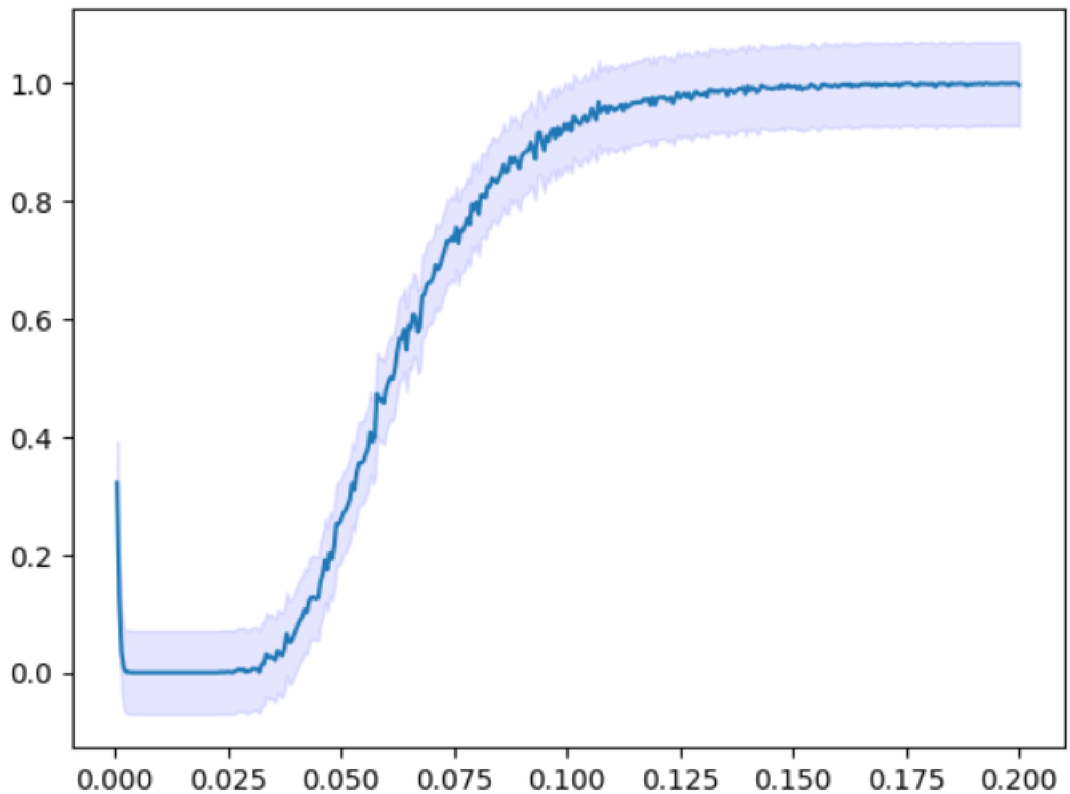

Fig. 5. Simulation for the 3D network

IV. Simulations

Figure 5 denotes the simulated values of the connectivity probability of the $9 \times 9 \times 9$ 3D network for $\theta \in [0, 0.2]$. Although it seems like the connectivity probability steadily increases from 0 to 1, we believe that the 3D network case also exhibits a phase transition at $\theta = 1/26$ similar to the 2D case. It is hard to see this in simulation, particularly for values of θ close to $1/26$. For values of θ close to $1/26$ it may take a long time for the connectivity probability to approach 1, hence we may need to simulate large networks which is infeasible with the available computational power.

V. Conclusion

We focussed on the problem of finding the asymptotic connectivity probability of cell partitioned networks. We considered 2D and 3D square and cubic networks, where we fully characterized the relationship between the asymptotic connectivity probability and the cell density for

2D networks, and we partially characterize the above relationship, for 3D networks. Full/partial characterization of a general graph, considering combinations of edge and site percolation models, simulating real world networks with real world data and obtaining tight connection probability bounds for networks of given size can be possible future work.

REFERENCES

[1] M. Franceschetti and R. Meester, *Random Networks for Communication: From Statistical Physics to Information Systems*, ser. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2008.

[2] J. Spencer, *Ten Lectures on the Probabilistic Method*, 2nd ed. Society for Industrial and Applied Mathematics, 1994. [Online]. Available: https://epubs.siam.org/doi/abs/10.1137/1.9781611970074

[3] B. Krishnamachari, S. B. Wicker, R. Béjar, and M. R. Pearlman, “Critical density thresholds in distributed wireless networks,” 2003. [Online]. Available: https://api.semanticscholar.org/CorpusID:13099279

[4] M. J. Neely, “Ee 550: Notes on capacity and connectivity in large networks,” 2024. [Online]. Available: https://ee.usc.edu/stochastic-nets/docs/notes-capacity-large-networks.pdf

[5] M. Frodigh, P. A. Johansson, and P. Larsson, “Wireless ad hoc networking—the art of networking without a network,” 2000. [Online]. Available: https://api.semanticscholar.org/CorpusID:7814534

[6] S. Gambhir, N. Aneja, and S. Mangla, “Need of ad-hoc social network based on users’ dynamic interests,” in *2015 International Conference on Soft Computing Techniques and Implementations (ICSCTI)*, 2015, pp. 52–56.

[7] F. Gulec, B. Atakan, and F. Dressler, “Mobile human ad hoc networks: A communication engineering viewpoint on interhuman airborne pathogen transmission,” *Nano Communication Networks*, vol. 32-33, p. 100410, 2022. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S187877892200014X

[8] A. Barbour, L. Holst, and S. Janson, *Poisson Approximation*, ser. Oxford studies in probability. Clarendon Press, 2023.